ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\left(\sin\frac{A}{2}\right)^{\sin\frac{A}{2}}+\left(\sin\frac{B}{2}\right)^{\sin\frac{B}{2}}+\left(\sin\frac{C}{2}\right)^{\sin\frac{C}{2}}\ge\frac{3\sqrt{2}}{2}$$

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$$\text{LHS}\overset{A-G}{\ge}3\prod_{cyc}\left(\left(\sin\frac{A}{2}\right)^{\frac{\sin\frac{A}{2}}{3}}\right)\overset{?}{\ge}\frac{3\sqrt{2}}{2}\Leftrightarrow\sum_{cyc}\left(\frac{\sin\frac{A}{2}}{3}.\ln\left(\sin\frac{A}{2}\right)\right)\overset{?}{\ge}\ln\frac{1}{\sqrt{2}}$$

$$\Leftrightarrow\sum_{cyc}\left(\sin\frac{A}{2}.\ln\left(\sin\frac{A}{2}\right)\right)\underbrace{\overset{?}{\ge}}_{(*)}3\ln\frac{1}{\sqrt{2}}$$

$$\text{Let } f(x)=\sin\frac{x}{2}.\ln\left(\sin\frac{x}{2}\right)\ \forall x\in\left(0,\frac{\pi}{2}\right)\text{ and then :}$$

$$f''(x)=\frac{1-\left(\sin^2\frac{x}{2}\right)\left(2+\ln\left(\sin\frac{x}{2}\right)\right)}{4\sin\frac{x}{2}}\to(1)$$

$$\because\ln\left(\sin\frac{x}{2}\right)\le\sin\frac{x}{2}-1\ \therefore\left(\sin^2\frac{x}{2}\right)\left(2+\ln\left(\sin\frac{x}{2}\right)\right)\le\left(\sin^2\frac{x}{2}\right)\left(1+\sin\frac{x}{2}\right)\overset{?}{<}1$$

$$\Leftrightarrow\csc^2\frac{x}{2}\overset{?}{>}1+\sin\frac{x}{2}\Leftrightarrow\cot^2\frac{x}{2}\overset{?}{>}\sin\frac{x}{2}\Leftrightarrow\cos^2\frac{x}{2}\overset{?}{>}\sin^3\frac{x}{2}\to\text{true}\because 0<\frac{x}{2}<\frac{\pi}{4}$$

$$\Rightarrow\cos^2\frac{x}{2}>\sin^2\frac{x}{2}\overset{1>\sin\frac{x}{2}}{>}\sin^3\frac{x}{2}\ \therefore\left(\sin^2\frac{x}{2}\right)\left(2+\ln\left(\sin\frac{x}{2}\right)\right)<1\overset{\text{via }(1)}{\Rightarrow}f''(x)>0$$

$$\therefore\sum_{cyc}\left(\sin\frac{A}{2}.\ln\left(\sin\frac{A}{2}\right)\right)\overset{\text{Jensen}}{\ge}3\sin\frac{\pi}{6}.\ln\left(\sin\frac{\pi}{6}\right)=\frac{3}{2}.\ln\frac{1}{2}=3\ln\frac{1}{\sqrt{2}}\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\sin\frac{A}{2}\right)^{\sin\frac{A}{2}}+\left(\sin\frac{B}{2}\right)^{\sin\frac{B}{2}}+\left(\sin\frac{C}{2}\right)^{\sin\frac{C}{2}}\ge\frac{3\sqrt{2}}{2}$$

$\forall$ acute Δ ABC, " = " iff Δ ABC is equilateral (QED)