ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{h_a}{b+c}\sin A \le \frac{9}{8}$$

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Solution by Tapas Das-India

$$\sum \frac{1}{a} = \frac{\sum ab}{abc} \le \frac{(a+b+c)^2}{3abc} = \frac{4s^2}{12Rrs} = \frac{s}{3Rr} \overset{Doucet}{\le} \frac{4R+r}{3\sqrt{3}Rr} \overset{Euler}{\le} \frac{\frac{9R}{2}}{3\sqrt{3}Rr} = \frac{\sqrt{3}}{2r} \quad (1)$$

$$\sum \frac{h_a}{b+c}\sin A = \sum \frac{\frac{2F}{a}}{b+c}\left(\frac{a}{2R}\right) \le \sum \frac{F}{R}\cdot\frac{1}{b+c} \overset{AM-HM}{\le}$$

$$\le \frac{F}{4R}\sum\left(\frac{1}{b}+\frac{1}{c}\right) = \frac{F}{2R}\sum\frac{1}{a} \overset{(1)}{\le} \frac{F}{2R}\cdot\frac{\sqrt{3}}{2r} = \frac{1}{4}\cdot\frac{s\sqrt{3}}{R} \overset{Mitrinovic}{\le} \frac{1}{4}\cdot\frac{3\sqrt{3}\frac{R}{2}}{R}.\sqrt{3} = \frac{9}{8}$$

Equality holds for $a = b = c$