ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(h_a+h_b+h_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\geq 9$$

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Solution by Daniel Sitaru-Romania

$$(h_a+h_b+h_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)=\sum_{cyc}h_a\cdot\sum_{cyc}\frac{1}{r_a}=$$

$$=\sum_{cyc}\frac{2F}{a}\cdot\sum_{cyc}\frac{s-a}{F}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\cdot(s-a+s-b+s-c)=$$

$$=2\cdot\frac{ab+bc+ca}{abc}\cdot(3s-a-b-c)=2\cdot\frac{s^2+r^2+4Rr}{4RF}\cdot(3s-2s)\overset{GERRETSEN}{\geq}$$

$$\geq 2\cdot\frac{16Rr-5r^2+r^2+4Rr}{4Rrs}\cdot s=2\cdot\frac{20Rr-4r^2}{4Rr}=$$

$$=10-\frac{2r}{R}\overset{EULER}{\geq}10-2\cdot\frac{R}{2}\cdot\frac{1}{R}=10-1=9$$

Equality holds for $a=b=c$.