ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC. the following relationship holds :

$$(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2\sqrt{3}(m_a+m_b+m_c)$$

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$$2\sqrt{3}(m_a+m_b+m_c)\overset{\text{Chu and Yang}}{\leq}2\sqrt{3(4s^2-16Rr+5r^2)}\overset{?}{\leq}$$

$$(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{(s^2+4Rr+r^2)^2}{4Rrs}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^4-192R^2r^2s^2(4s^2-16Rr+5r^2)\underset{(*)}{\overset{?}{\geq}}0$$

$$and \because (s^2-16Rr+5r^2)^4\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*)\geq (s^2-16Rr+5r^2)^4 \Leftrightarrow (5R-6r)s^6-r(138R^2-63Rr+9r^2)s^4$$
$$+r^2(1232R^3-1008R^2r+303Rr^2-31r^3)s^2$$
$$-r^3(4080R^4-5136R^3r+2394R^2r^2-501Rr^3+39r^4)\overset{(**)}{\geq}0$$

$$and \because (5R-6r)(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**)\geq (5R-6r)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (51R^2-30Rr+3r^2)s^4-r(1304R^3-1080R^2r+276Rr^2-22r^3)s^2$$
$$+r^2(8200R^4-9080R^3r+3723R^2r^2-662Rr^3+43r^4)\overset{(***)}{\geq}0 \text{ and } \because$$

$$(51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0 \therefore \text{ in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***)\geq (51R^2-30Rr+3r^2)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (164R^3-195R^2r+60Rr^2-4r^3)s^2\overset{(****)}{\geq}$$
$$r(2428R^4-3380R^3r+1560R^2r^2-284Rr^3+16r^4)$$

$$\text{Now, LHS of } (****)\overset{\text{Gerretsen}}{\geq}(164R^3-195R^2r+60Rr^2-4r^3)(16Rr-5r^2)$$
$$\overset{?}{\geq}r(2428R^4-3380R^3r+1560R^2r^2-284Rr^3+16r^4)$$
$$\Leftrightarrow 196t^4-560t^3+375t^2-80t+4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(112t^3+84t^2(t-2)+39t-2)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$
$$\Rightarrow (****)\Rightarrow (***)\Rightarrow (**)\Rightarrow (*) \text{ is true} \therefore (ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
$$\geq 2\sqrt{3}(m_a+m_b+m_c)\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$