ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}} \ge \frac{3}{8}$$

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$$WLOG\ a \ge b \ge c\ then\ \ \sin\frac{A}{2} \ge \sin\frac{B}{2} \ge \sin\frac{C}{2}\ and$$

$$\sin\frac{A}{2}+\sin\frac{B}{2} \ge \sin\frac{A}{2}+\sin\frac{C}{2} \ge \sin\frac{C}{2}+\sin\frac{B}{2}$$

$$\frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}} = \sin^2\frac{A}{2}\cdot\frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\ (1)$$

$$\sum \sin^2\frac{A}{2} = 1-\frac{r}{2R} \overset{Euler}{\ge} 1-\frac{1}{4}=\frac{3}{4}\ (2)$$

$$\sum \frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}} \overset{1}{=} \sum\left(\sin^2\frac{A}{2}\cdot\frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\right) \overset{Chebyshev}{\ge}$$

$$\ge \frac{1}{3}\left(\sum \sin^2\frac{A}{2}\right)\left(\sum \frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\right) \overset{Nesbitt\ \&\ (2)}{\ge} \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{3}{2}=\frac{3}{8}$$

$$Equality\ holds\ \ for\ A=B=C$$