ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{b+c}sec\frac{A}{2}+\frac{b}{c+a}sec\frac{B}{2}+\frac{c}{a+b}sec\frac{C}{2}\geq\sqrt{3}$$

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Let be $f:(0,\pi)\to\mathbb{R}, f(x)=sec\frac{x}{2}$

$$f'(x)=\frac{1}{2}\cdot\frac{sin\frac{x}{2}}{cos^2\frac{x}{2}}\quad f''(x)=\frac{1}{2}\cdot\frac{\frac{1}{2}\cdot cos^3\frac{x}{2}+sin\frac{x}{2}\cdot 2\cdot\frac{1}{2}\cdot sin\frac{x}{2}cos\frac{x}{2}}{cos^4\frac{x}{2}}$$

$$f''(x)=\frac{1}{4}\cdot\frac{cos^2\frac{x}{2}+2sin^2\frac{x}{2}}{cos^3\frac{x}{2}}=\frac{1}{4}\cdot\frac{1+sin^2\frac{x}{2}}{cos^3\frac{x}{2}}$$

$x\in(0,\pi)\to\frac{x}{2}\in\left(0,\frac{\pi}{2}\right)\to sin\frac{x}{2}>0, cos\frac{x}{2}>0\to f''(x)>0\to f$ convex

By Jensen's inequality:

$$f(A)+f(B)+f(C)\geq 3f\left(\frac{A+B+C}{3}\right)$$

$$\sum_{cyc}sec\frac{A}{2}\geq 3sec\frac{\pi}{6}=\frac{3}{\frac{\sqrt{3}}{2}}=2\sqrt{3}\quad (1)$$

WLOG: $a\geq b\geq c\to$

$$\to b+c\leq a+c\leq a+b\to\frac{1}{b+c}\geq\frac{1}{a+c}\geq\frac{1}{a+b}\to$$

$$\frac{a}{b+c}\geq\frac{b}{a+c}\geq\frac{c}{a+b}\quad (2)$$

$$a\geq b\geq c\to\frac{A}{2}\geq\frac{B}{2}\geq\frac{C}{2}\to cos\frac{A}{2}\leq cos\frac{B}{2}\leq cos\frac{C}{2}\to$$

$$\to sec\frac{A}{2}\geq sec\frac{B}{2}\geq sec\frac{C}{2}\qquad (3)$$

By $(2),(3)$:

$$\sum_{cyc}\frac{a}{b+c}\cdot sec\frac{A}{2}\overset{CEBYSHEV}{\geq}\frac{1}{3}\cdot\sum_{cyc}\frac{a}{b+c}\cdot\sum_{cyc}sec\frac{A}{2}\overset{NESBITT}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}\cdot\sum_{cyc}sec\frac{A}{2}\overset{(1)}{\geq}\frac{1}{2}\cdot 2\sqrt{3}=\sqrt{3}$$

Equality holds for $a=b=c$.