# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$\frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)\geq 2\sqrt{3}$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)$$

$$+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=r_a,y=r_b,z=r_c,A'=\sec\frac{A}{2},B'=\sec\frac{B}{2},C'=\sec\frac{C}{2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B'+C'}^2+\frac{y}{z+x}.\sqrt{C'+A'}^2+\frac{z}{x+y}.\sqrt{A'+B'}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F'.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\sec\frac{A}{2}\sec\frac{B}{2}\right)}$$

$$=\sqrt{3\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}.\sum_{cyc}\cos\frac{A}{2}}=\sqrt{\frac{6R}{s}.\sum_{cyc}\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}}\overset{0<\cos\frac{B-C}{2}\leq 1\text{ and analogs}}{\geq}$$

$$\sqrt{\frac{6R}{s}\cdot\sum_{cyc}(\sin B+\sin C)}=\sqrt{\frac{12R}{s}\cdot\frac{s}{R}}=2\sqrt{3}$$

$$\therefore \frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)$$
$$\geq 2\sqrt{3}\ \forall\ \Delta\ ABC,''=''\ \text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$