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In any Δ ABC, the following relationship holds :

$$\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\le 2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)=2\sum_{cyc}\frac{s-b}{s-a}=2\sum_{cyc}\frac{(s-b)^2}{(s-a)(s-b)}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{2\left(\sum_{cyc}(s-a)\right)^2}{\sum_{cyc}\left((s-a)(s-b)\right)}=\frac{2s^2}{4Rr+r^2}\overset{?}{\ge}\sum_{cyc}\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\sum_{cyc}\frac{\frac{(b+c)}{a}\sin\frac{A}{2}}{\sin\frac{A}{2}}$$

$$=\frac{1}{4Rrs}\cdot\sum_{cyc}\left(bc(2s-a)\right)=\frac{2s}{4Rrs}\cdot(s^2+4Rr+r^2-6Rr)$$

$$\Leftrightarrow\frac{2s^2}{4R+r}\overset{?}{\ge}\frac{s^2-2Rr+r^2}{2R}\Leftrightarrow s^2\overset{?}{\le}8R^2-2Rr-r^2$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-(4R^2-6Rr-4r^2)\overset{?}{\le}0$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(2R+r)(R-2r)\overset{?}{\le}0\to\text{true}$$

$$\because s^2-(4R^2+4Rr+3r^2)\overset{\text{Gerretsen}}{\le}0\text{ and }-2(2R+r)(R-2r)\overset{\text{Euler}}{\le}0$$

$$\therefore\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\le 2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)