ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \cos A + 2\sum \tan^2\frac{A}{2} \ge \frac{7}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum \cos A + 2\sum \tan^2\frac{A}{2} = \left(1+\frac{r}{R}\right) + 2\left(\left(\frac{4R+r}{s}\right)^2 - 2\right)$$

$$\textit{We need to show: } \left(1+\frac{r}{R}\right) + 2\left(\left(\frac{4R+r}{s}\right)^2 - 2\right) \ge \frac{7}{2} \ or$$

$$2\left(\frac{4R+r}{s}\right)^2 + \frac{r}{R} - 3 \ge \frac{7}{2} \ or$$

$$2\left(\frac{4R+r}{s}\right)^2 + \frac{r}{R} \ge \frac{7}{2} + 3 \ or$$

$$2\left(\frac{4R+r}{s}\right)^2 + \frac{r}{R} \ge \frac{13}{2} \ or$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2} + \frac{r}{R} \ge \frac{13}{2} (Gerretsen) \ or$$

$$\frac{2R(16R^2+8Rr+r^2) + r(4R^2+4Rr+3r^2)}{(4R^2+4Rr+3r^2)R} \ge \frac{13}{2} or$$

$$(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3) \ge$$

$$13(4R^2+4Rr+3r^2)R \ or \ (64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3) \ge$$
$$\ge 52R^3+52R^2r+39Rr^3 \ or \ (64R^3+40R^2r+12Rr^3+6r^3) - (52R^3+52R^2r+39Rr^3)$$

$$12R^3 - 12R^2r - 27Rr^2 + 6r^3 \ge 0 \ or$$

$$(R-2r)(12R^2+12Rr-3r^2) \ge 0 \ true \ (Euler)$$

$$\textit{Equality holds for } A = B = C$$