ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \cos A + 2\sum \sec^2\frac{A}{2} \geq \frac{19}{2}$$

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$$\sum \cos A + 2\sum \sec^2\frac{A}{2} = \left(1+\frac{r}{R}\right) + 2\left(3+\sum \tan^2\frac{A}{2}\right) =$$

$$= \left(1+\frac{r}{R}\right) + 6 + 2\left(\left(\frac{4R+r}{s}\right)^2 - 2\right) = 3 + 2\left(\frac{4R+r}{s}\right)^2 + \frac{r}{R} \overset{Gerretsen}{\geq}$$

$$\geq\ 3 + \frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2} + \frac{r}{R}$$

$$We\ need\ to\ show: 3 + \frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2} + \frac{r}{R} \geq \frac{19}{2}$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2} + \frac{r}{R} \geq \frac{19}{2} - 3\ \ or$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2} + \frac{r}{R} \geq \frac{13}{2}$$

$$\frac{2R(16R^2+8Rr+r^2) + r(4R^2+4Rr+3r^2)}{(4R^2+4Rr+3r^2)R} \geq \frac{13}{2}$$

$$(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3) \geq 13(4R^2+4Rr+3r^2)R\ or$$

$$\left(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3\right) \geq$$
$$\geq 52R^3+52R^2r+39Rr^3\ or\left(64R^3+40R^2r+12Rr^3+6r^3\right) - (52R^3+52R^2r+39Rr^3)$$

$$12R^3 - 12R^2r - 27Rr^2 + 6r^3 \geq 0$$

$$(R-2r)(12R^2+12Rr-3r^2) \geq 0\ true\ (Euler)$$

$$Equality\ holds\ for\ A = B = C$$