ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \sin\frac{A}{2}+\sum \cot^2\frac{A}{2}\geq \frac{21}{2}$$

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$$\sum \sin\frac{A}{2}\overset{Jensen}{\leq} 3\sin\frac{\pi}{6}=\frac{3}{2}\ \left(f(x)=\sin x\ \text{is concave in}\ \left(0,\frac{\pi}{2}\right)\right)\ (1)$$

$$\text{We need to show}\sum \sin\frac{A}{2}+\sum \cot^2\frac{A}{2}\geq \frac{21}{2}\ \text{or}$$

$$\sum \sin\frac{A}{2}+\sum cosec^2\frac{A}{2}-3\geq \frac{21}{2}\ \text{or}$$

$$\sum \sin\frac{A}{2}+\sum cosec^2\frac{A}{2}\geq \frac{21}{2}+3=\frac{27}{2}\ \text{or}$$

$$\sum \sin\frac{A}{2}+\sum \frac{1}{\sin^2\frac{A}{2}}\geq \frac{27}{2}\ (2)$$

$$\text{let}\ \sin\frac{A}{2}=x,\sin\frac{B}{2}=y,\sin\frac{C}{2}=z$$

$$\text{Using (1) and (2) we need to show}\sum x+\sum \frac{1}{x^2}\geq \frac{27}{2}$$

$$\text{where}\ x+y+z\leq \frac{3}{2}\ \text{and}\ x,y,z\in(0,1)$$

$$\text{we will show}\ x+\frac{1}{x^2}\geq 12-15x\ (3)$$

$$\text{proof}: x+\frac{1}{x^2}\geq 12-15x\ \text{or}\ x^3+1\geq 12x^2-15x^3\ \text{or}$$

$$16x^3-12x^2+1\geq 0\ \text{or}\ (2x-1)^2(4x+1)\geq 0\ \text{true as}\ x\in(0,1)$$

$$\sum\left(x+\frac{1}{x^2}\right)\overset{(3)}{\geq}\sum(12-15x)=3\times 12-15(x+y+z)=$$

$$=36-15(x+y+z)\geq 36-15\times\frac{3}{2}=\frac{27}{2}\left(\text{as}\ \ x+y+z\leq\frac{3}{2}\right)\ \text{or}$$

$$\sum x+\sum\frac{1}{x^2}\geq\frac{27}{2}$$

$$\text{Equality holds for}\ x=y=z=\frac{1}{2}\ \text{or}\ \sin\frac{A}{2}=\sin\frac{B}{2}=\sin\frac{C}{2}=\frac{1}{2}\ \text{or}\ A=B=C=\frac{\pi}{3}$$