ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$a^{r_a}b^{r_b}c^{r_c} \geq \left(2\sqrt{3}r\right)^{9r}$$

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$$WLOG\ a \geq b \geq c\ then\ r_a \geq r_b \geq r_c$$

$$\sum \frac{r_a}{a} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(r_a+r_b+r_c) \overset{Leunberger\ II}{\leq} \frac{1}{3}(4R+r)\frac{\sqrt{3}}{2r} = \frac{4R+r}{2\sqrt{3}r} \quad (1)$$

$$We\ know\ that \sum \frac{1}{r_a} = \frac{1}{r}, \qquad \frac{\sum r_a}{3} \overset{AM-HM}{\geq} \frac{3}{\sum \frac{1}{r_a}} = 3r\ or$$

$$\sum r_a \geq 9r \quad (2)$$

$$Let\ us\ consider\ a\ with\ associated.weight\ r_a, b\ with\ r_b\ and\ c\ with\ r_c$$

$$GM \geq HM\ or\left(a^{r_a}b^{r_b}c^{r_c}\right)^{\frac{1}{(r_a+r_b+r_c)}} \geq \frac{(r_a+r_b+r_c)}{\sum \frac{r_a}{a}} \overset{(1)}{\geq} \frac{4R+r}{\frac{4R+r}{2\sqrt{3}r}} = \left(2\sqrt{3}r\right)$$

$$a^{r_a}b^{r_b}c^{r_c} \geq \left(2\sqrt{3}r\right)^{(r_a+r_b+r_c)} \overset{(2)}{\geq} \left(2\sqrt{3}r\right)^{9r}$$

$$Equality\ holds\ for\ a = b = c$$