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In ΔABC the following relationship holds:

$$\sum \frac{\csc^2 \frac{A}{2}}{\cot \frac{B}{2} + \cot \frac{C}{2}} \geq 2\sqrt{3}$$

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Solution by Tapas Das-India

$$\cot \frac{B}{2} + \cot \frac{C}{2} = \frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} = \frac{\cos \frac{B}{2} \sin \frac{C}{2} + \cos \frac{C}{2} \sin \frac{B}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} =$$

$$= \frac{\sin \frac{B+C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} \overset{A+B+C=\pi}{=} \frac{\cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} \ (1)$$

$$\sum \frac{\csc^2 \frac{A}{2}}{\cot \frac{B}{2} + \cot \frac{C}{2}} \overset{(1)}{=} \sum \frac{\csc^2 \frac{A}{2}}{\frac{\cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}} = \sum \frac{\csc^2 \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \overset{AM-GM}{\geq}$$

$$\geq 3 \sqrt[3]{\sec \frac{A}{2}} = 3 \sqrt[3]{\frac{4R}{s}} \overset{Mitrinovic}{\geq} 3 \sqrt[3]{\frac{4R}{\frac{3\sqrt{3}R}{2}}} = 3.\frac{2}{\sqrt{3}} = 2\sqrt{3}$$

$(Equality\ \ holds\ for\ A = B = C)$