ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \geq 2\sqrt{3}$$

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Solution by Tapas Das-India

$$\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}=\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=\frac{\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{\sin\frac{\pi-A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\ (1)$$

$$\frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\overset{(1)}{=}\frac{1}{\cos^2\frac{A}{2}\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}=\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^3\frac{A}{2}}\ (2)$$

$$\sum\frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\overset{(2)}{=}\sum\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^3\frac{A}{2}}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{1}{\prod\cos\frac{A}{2}}}=$$

$$=3\sqrt[3]{\frac{4R}{s}}\overset{Mitrinovic}{\geq}3\sqrt[3]{\frac{4R}{\frac{3\sqrt{3}R}{2}}}=3.\sqrt[3]{\frac{8}{3\sqrt{3}}}=\frac{3.2}{\sqrt{3}}=2\sqrt{3}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$