ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3}{2} < \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} < \frac{4\pi}{5}$$

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Solution by Tapas Das-India

$$We\ know\ that\ in\ \Delta ABC\ , b+c > a\ or\ 2(b+c) > a+b+c\ or$$

$$b+c > \frac{1}{2}(a+b+c)$$

$$similarly, (a+b) > \frac{1}{2}(a+b+c)\ and\ c+a > \frac{1}{2}(a+b+c)$$

$$\sum \frac{a}{b+c} < \sum \frac{a}{\frac{1}{2}(a+b+c)} = \sum \frac{2a}{a+b+c} = 2 \quad (1)$$

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \overset{CBS}{<} \sqrt{3\sum \frac{a}{b+c}} \overset{(1)}{<} \sqrt{3.2} = \sqrt{6}$$

$$We\ need\ to\ show$$

$$\sqrt{6} < \frac{4\pi}{5}\ or\ 5\sqrt{6} < 4 \times \frac{22}{7}\ or, 35\sqrt{6} < 88\ or,$$

$$1225 \times 6 \overset{squaring}{<} 88 \times 88\ or, 7350 < 7744\ (True)$$

$$Now\ \ a < b+c, so\ \frac{a}{b+c} < 1\ , we\ \ can\ say\ \sqrt{\frac{a}{b+c}} > \frac{a}{b+c}$$

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \overset{Nestbitt}{>} \frac{3}{2}$$