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In ΔABC the following relationship holds:

$$(a+b+c)\left(\frac{1}{r_a+r_b}+\frac{1}{r_b+r_c}+\frac{1}{r_c+r_a}\right)\ge 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\text{According to the known formula}:\ \boxed{r_a=r\,.\,\tan\frac{A}{2}}$$

$$r_a+r_b=r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=r\,.\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\,.\,\cos\frac{B}{2}}=r\,.\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\,.\,\cos\frac{B}{2}}=\frac{r\,.\,\cos^2\frac{C}{2}}{\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\cos\frac{C}{2}}$$

$$\text{Then }\sum_{cyc}\frac{1}{r_a+r_b}=\frac{1}{r}\,.\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\cos\frac{C}{2}\,\Big(\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}}\Big)=\frac{1}{r}\,.\,\left(\frac{P}{4R}\right).\,\Big(\sum_{cyc}\Big(1+\tan^2\frac{A}{2}\Big)\Big)=$$

$$\frac{1}{r}\,.\,\left(\frac{P}{4R}\right).\Big(3+\sum_{cyc}\tan^2\frac{A}{2}\Big)\overset{\boxed{\cos\frac{A}{2}.\,\cos\frac{B}{2}.\cos\frac{C}{2}=\frac{P}{4R}}(true)\ ,\ \boxed{\sum_{cyc}\tan^2\frac{A}{2}\ge 1}(true)}{\ge}\frac{1}{r}\,.\,\frac{P}{4R}\,.\,(3+1)=\frac{P}{Rr}$$

$$(a+b+c)\,.\sum_{cyc}\frac{1}{r_a+r_b}\ge 2p\,.\,\frac{P}{Rr}=2p^2\,.\,\frac{2(a+b+c)}{abc}\overset{\boxed{Rr=\frac{abc}{2(a+b+c)}}}{=}\frac{8p^3}{abc}=$$

$$=\frac{(2p)^3}{abc}\ge\frac{\left(3\sqrt[3]{abc}\right)^3}{abc}=27$$

$$\text{Equality holds iff } a=b=c$$