# ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$\frac{\tan A+\tan B+\tan C}{\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}}}\ge\sqrt[6]{243}$$

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$$\textit{For acute triangle}\ \cos(A-B)\le 1\ (1)$$

$$(\tan A+\tan B)=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}=$$

$$=\frac{2\sin(A+B)}{2\cos A\cos B}\overset{A+B+C=\pi}{=}\frac{2\sin C}{\cos(A+B)+\cos(A-B)}\overset{(1)}{\ge}\frac{2\sin C}{\cos(A+B)+1}=$$

$$\overset{A+B+C=\pi}{=}\frac{2\sin C}{1-\cos C}=\frac{4\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin^2\frac{C}{2}}=2\cot\frac{C}{2},$$

$$\textit{similarly},(\tan C+\tan B)\ge 2\cot\frac{A}{2},\quad(\tan C+\tan A)\ge 2\cot\frac{B}{2}$$

$$\textit{Using above result we get}\sum\tan A\ge\sum\cot\frac{A}{2}\quad(3)$$

$$\sum\cot\frac{A}{2}\cot\frac{B}{2}\overset{\forall x\,y,z>0\ 3\sum xy\le(\sum x)^2}{\le}\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3}\quad(4)$$

$$\therefore\frac{\tan A+\tan B+\tan C}{\sqrt[3]{\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}}}\overset{(4)\&(3)}{\ge}\frac{\sum\cot\frac{A}{2}}{\sqrt[3]{\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3}}}=\sqrt[3]{3}\left(\sum\cot\frac{A}{2}\right)^{\frac{1}{3}}=$$

$$=\sqrt[3]{3}\left(\frac{s}{r}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\ge}\sqrt[3]{3}(3\sqrt{3})^{\frac{1}{3}}=3^{\frac{1}{2}}.\sqrt[3]{3}=\sqrt[6]{3^5}=\sqrt[6]{243}$$

$$\textit{Equality holds for}\ A=B=C=\frac{\pi}{3}$$