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In ΔABC the following relationship holds:

$$\frac{\csc A+\csc B+\csc C}{\sqrt[3]{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}}\geq 2\sqrt[3]{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\csc A=\sum\frac{1^2}{\sin A}\overset{Bergstom}{\geq}\frac{(1+1+1)^2}{\sum\sin A}=\frac{9}{\frac{s}{R}}=\frac{9R}{s}=\sqrt[3]{\frac{9^3R^3}{s^3}}\ (1)\ and$$

$$\sum\tan\frac{A}{2}=\frac{4R+r}{s}\ (2)$$

$$\frac{\csc A+\csc B+\csc C}{\sqrt[3]{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}}=\frac{\sum\csc A}{\sqrt[3]{\sum\tan\frac{A}{2}}}\overset{(1)\&(2)}{\geq}\frac{\sqrt[3]{\frac{9^3R^3}{s^3}}}{\sqrt[3]{\frac{4R+r}{s}}}=$$

$$=\sqrt[3]{\frac{9^3R^3}{s^2(4R+r)}}\overset{Mitrinovic\ \&\ Euler}{\geq}\sqrt[3]{\frac{9^3R^3}{\frac{27R^2}{4}\left(4R+\frac{R}{2}\right)}}=\sqrt[3]{\frac{9^3R^3}{\frac{27R^2}{4}\left(\frac{9R}{2}\right)}}=\sqrt[3]{8\times 3}=2\sqrt[3]{3}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$