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In ΔABC the following relationship holds:

$$\frac{2F}{R}\le(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}\le\frac{F}{r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2}\overset{Jensen}{\le}3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}\ (1)$$

$$(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}\overset{Chebyshev}{\le}$$

$$\le\frac{1}{3}\left(\sum(b+c-a)\right)\left(\sum\sin\frac{A}{2}\right)\le 2s.\frac{3}{2}.\frac{1}{3}=s=\frac{sr}{r}=\frac{F}{r}$$

$$(b+c-a)\sin\frac{A}{2}+(c+a-b)\sin\frac{B}{2}+(a+b-c)\sin\frac{C}{2}=$$

$$=\sum(b+c-a)\sin\frac{A}{2}=\sum 2(s-a)\sin\frac{A}{2}\overset{AM-GM}{\ge}6\sqrt[3]{\prod(s-a)\sin\frac{A}{2}}=$$

$$=6\left(\frac{sr^2r}{4R}\right)^{\frac{1}{3}}=6\left(\frac{s^3r^3}{4Rs^2}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\ge}6\left(\frac{s^3r^3}{4R\frac{27}{4}R^2}\right)^{\frac{1}{3}}=\frac{6rs}{3R}=\frac{2F}{R}$$

$$Equality\ holds\ for\ a=b=c$$