ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cot A+\cot B+\cot C+\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\geq 2(\csc A+\csc B+\csc C)$$

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$$\textit{We know that in } \Delta ABC:\sum\tan A\tan B=1+\frac{4R^2}{s^2-(2R+r)^2}$$

$$\prod\tan A=\frac{2sr}{s^2-(2R+r)^2}$$

$$\sum\sin A\sin B=\frac{s^2+r^2+4Rr}{4R^2},\qquad \prod\sin A=\frac{sr}{2R^2}$$

Using the above result we get:

$$\sum\csc A=\sum\frac{1}{\sin A}=\frac{\sum\sin A\sin B}{\prod\sin A}=\frac{s^2+r^2+4Rr}{2sr}\ \textit{and}$$

$$\sum\cot A=\sum\frac{1}{\tan A}=\frac{\sum\tan A\tan B}{\prod\tan A}=\frac{s^2-(2R+r)^2+4R^2}{2sr},$$

$$\prod\cot\frac{A}{2}=\frac{s}{r}$$

We need to show:

$$\cot A+\cot B+\cot C+\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\geq 2(\csc A+\csc B+\csc C)$$

$$\frac{s^2-(2R+r)^2+4R^2}{2sr}+\frac{s}{r}\geq\frac{s^2+r^2+4Rr}{2sr}\ \textit{or}$$

$$s^2-(2R+r)^2+4R^2+2s^2-2s^2-2r^2-8Rr\geq 0\ \textit{or}$$

$$16Rr-5r^2-4R^2-4Rr-r^2+4R^2-2r^2-8Rr\geq 0\ (\textit{Gerretsen})\ \textit{or}$$

$$4Rr-8r^2\geq 0\ \textit{or}\ \ R\geq 2r\ \ \textit{true Euler}$$

$$\textit{Equality holds for } A=B=C$$