ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\cos A\cos B\cos C \le \frac{1}{8}\cos(A-B)\cos(B-C)\cos(C-A)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos(A-B)\cos(B-C)\cos(C-A)$$
$$=\left(2\cos^2\frac{A-B}{2}-1\right)\left(2\cos^2\frac{B-C}{2}-1\right)\left(2\cos^2\frac{C-A}{2}-1\right)$$
$$=8\prod_{cyc}\cos^2\frac{B-C}{2}-4\left(\prod_{cyc}\cos^2\frac{B-C}{2}\right)\left(\sum_{cyc}\sec^2\frac{B-C}{2}\right)+2\sum_{cyc}\cos^2\frac{B-C}{2}-1\to(a)$$

$$\text{Now, }\sum_{cyc}\cos^2\frac{B-C}{2}=\sum_{cyc}\frac{(b+c)^2\sin^2\frac{A}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}=\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc(b+c)^2}{s-a}$$
$$=\frac{1}{16R^2s}\cdot\sum_{cyc}\frac{bc(s+s-a)^2}{s-a}=\frac{1}{16R^2s}\cdot\sum_{cyc}\left(\frac{bcs^2}{s-a}+2sbc+bc(s-a)\right)$$
$$=\frac{1}{16R^2s}\left(s^3\sum_{cyc}\sec^2\frac{A}{2}+3s\sum_{cyc}ab-3abc\right)$$
$$=\frac{1}{16R^2s}\left(s^3\left(\frac{s^2+(4R+r)^2}{s^2}\right)+3s(s^2+4Rr+r^2)-12Rrs\right)$$
$$=\frac{4s^2+(4R+r)^2+3r^2}{16R^2}\Rightarrow\sum_{cyc}\cos^2\frac{B-C}{2}=\frac{4s^2+(4R+r)^2+3r^2}{16R^2}\to(1)$$

$$\text{Again, }\sum_{cyc}\sec^2\frac{B-C}{2}=\sum_{cyc}\frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{(b+c)^2\sin^2\frac{A}{2}}=\sum_{cyc}\frac{16R^2s(s-a)a}{4Rrs(b+c)^2}$$
$$=\frac{2R}{r}\sum_{cyc}\frac{a(b+c-a)}{(b+c)^2}=\frac{2R}{r}\left(\sum_{cyc}\frac{a}{b+c}-\sum_{cyc}\frac{a^2}{(b+c)^2}\right)\to(2)$$

$$\text{Now, }\sum_{cyc}\frac{a}{b+c}=\frac{\sum_{cyc}(a(c+a)(a+b))}{\prod_{cyc}(b+c)}=\frac{\sum_{cyc}\left(a(\sum_{cyc}ab+a^2)\right)}{2s(s^2+2Rr+r^2)}$$
$$=\frac{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)}=\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}\to(3)$$

$$\text{and, }\sum_{cyc}\frac{a^2}{(b+c)^2}=\sum_{cyc}\frac{\left(2s-(b+c)\right)^2}{(b+c)^2}=\sum_{cyc}\frac{4s^2-4s(b+c)+(b+c)^2}{(b+c)^2}$$
$$\overset{(i)}{=}4s^2\left(\frac{\sum_{cyc}((c+a)^2(a+b)^2)}{\left(\prod_{cyc}(b+c)\right)^2}\right)-4s\left(\frac{\sum_{cyc}(c+a)(a+b)}{\prod_{cyc}(b+c)}\right)+3$$

$$\text{We have : }\sum_{cyc}((c+a)^2(a+b)^2)=\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)^2$$

$$=\sum_{cyc}\left(\left(\sum_{cyc}ab\right)^2+2a^2\sum_{cyc}ab+a^4\right)$$

$$=3\left(\sum_{cyc}ab\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2$$

$$=\left(\sum_{cyc}ab\right)^2+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}a^2\right)^2+2\sum_{cyc}a^2b^2+4abc(2s)$$

$$-2\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab+\sum_{cyc}a^2\right)^2+32Rrs^2=\left(3s^2-4Rr-r^2\right)^2+32Rrs^2\to(ii)$$

$$\text{Again,}\sum_{cyc}(c+a)(a+b)=\sum_{cyc}\left(\sum_{cyc}ab+a^2\right)=3\sum_{cyc}ab+\sum_{cyc}a^2$$

$$=\sum_{cyc}a^2+2\sum_{cyc}ab+\sum ab=4s^2+s^2+4Rr+r^2$$

$$\therefore\sum_{cyc}(c+a)(a+b)=5s^2+4Rr+r^2\to(iii)$$

$$\because\prod_{cyc}(b+c)=s^2+2Rr+r^2\ \therefore\ (i),(ii),(iii)\Rightarrow\sum_{cyc}\frac{a^2}{(b+c)^2}=$$

$$\frac{4s^2\left(\left(3s^2-4Rr-r^2\right)^2+32Rrs^2\right)}{4s^2(s^2+2Rr+r^2)^2}-\frac{4s\left(5s^2+4Rr+r^2\right)}{2s(s^2+2Rr+r^2)}+3$$

$$=\frac{\left(3s^2-4Rr-r^2\right)^2+32Rrs^2-2\left(5s^2+4Rr+r^2\right)\left(s^2+2Rr+r^2\right)+3\left(s^2+2Rr+r^2\right)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\Rightarrow$$

$$\sum_{cyc}\frac{a^2}{(b+c)^2}=\frac{2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\to(4);$$

$$(2),(3),(4)\Rightarrow\sum\sec^2\frac{B-C}{2}=$$

$$\frac{2R}{r}\left(\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}-\frac{2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\right)$$

$$=\frac{2R}{r}\left(\frac{\left(2s^2-2Rr-2r^2\right)\left(s^2+2Rr+r^2\right)-\left(2s^4-s^2\left(8Rr+12r^2\right)+12R^2r^2+8Rr^3+2r^4\right)}{(s^2+2Rr+r^2)^2}\right)$$

$$\to(5)$$

$$\text{Also, }8\prod_{cyc}\cos^2\frac{B-C}{2}=8\prod_{cyc}\frac{(b+c)^2\sin^2\frac{A}{2}}{a^2}=8\left(\frac{4s^2\left(s^2+2Rr+r^2\right)^2}{16R^2r^2s^2}\right)\left(\frac{r^2}{16R^2}\right)$$

$$=\frac{\left(s^2+2Rr+r^2\right)^2}{8R^4}\to(6)\ \therefore\ (a),(1),(5),(6)\Rightarrow\cos(A-B)\cos(B-C)\cos(C-A)$$

$$=\frac{\left(s^2+2Rr+r^2\right)^2}{8R^4}$$

$$-\left(\frac{(s^2+2Rr+r^2)^2}{16R^4}\right)\cdot\frac{2R}{r}\left(\frac{\begin{array}{c}(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\\(2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4)\end{array}}{(s^2+2Rr+r^2)^2}\right)$$

$$+\frac{4s^2+(4R+r)^2+3r^2}{8R^2}-1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A)$$

$$=\frac{r(s^2+2Rr+r^2)^2-R\sigma+R^2r(4s^2+(4R+r)^2+3r^2)-8R^4r}{8R^4r}$$

$$\left(\begin{array}{c}\text{where } \sigma=(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\\(2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4)\end{array}\right)$$

$$\geq 8\cos A\cos B\cos C=\frac{2(s^2-4R^2-4Rr-r^2)}{R^2}$$

$$\Leftrightarrow s^4-(22R^2+8Rr-2r^2)s^2+72R^4+88R^3r+38R^2r^2+8Rr^3+r^4 \boxed{\overset{(*)}{\geq}} 0 \text{ and}$$

$$\because (s^2-4R^2-4Rr-3r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (s^2-4R^2-4Rr-3r^2)^2$$

$$\Leftrightarrow (7R^2-4r^2)s^2 \boxed{\overset{(**)}{\leq}} 28R^4+28R^3-R^2r^2-8Rr^3-4r^4$$

$$\text{Again, } (7R^2-4r^2)s^2 \overset{\text{Rouche}}{\leq} (7R^2-4r^2)\left(\begin{array}{c}2R^2+10Rr-r^2\\+2(R-2r).\sqrt{R^2-2Rr}\end{array}\right)$$

$$\overset{?}{\leq} 28R^4+28R^3-R^2r^2-8Rr^3-4r^4$$

$$\Leftrightarrow (7R^2-4r^2)(R-2r).\sqrt{R^2-2Rr} \boxed{\underset{(***)}{\overset{?}{\leq}}} (R-2r)(7R^3-7R^2r-7Rr^2+2r^3) \text{ and}$$

$$\because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove :}$$

$$7R^3-7R^2r-7Rr^2+2r^3 > (7R^2-4r^2).\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (7R^3-7R^2r-7Rr^2+2r^3)^2 > (R^2-2Rr)(7R^2-4r^2)^2$$

$$\Leftrightarrow r^2(7R^4+14R^3+5R^2r^2+4Rr^3+4r^4) > 0 \to \text{true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore \cos A\cos B\cos C \leq \frac{1}{8}\cos(A-B)\cos(B-C)\cos(C-A)$$

$$\forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$case1$$

$$For\ acute\ triangle, 2\sin A\cos(B-C)=\sin 2B+\sin 2C \geq 2\sqrt{\sin 2B\sin 2C}$$

$$=4\sqrt{\sin B\sin C\,\cos B\cos C},$$

$$now \prod 2\sin A\cos(B-C) \geq 64\prod \sin A\prod \cos A\ or, \frac{1}{8}\prod \cos(B-C)$$

$$\geq \prod \cos A,$$

$$case2. for\ non\ acute \prod \cos A < 0 < \frac{1}{8}\prod \cos(B-C),\ equality\ A=B=C=\frac{\pi}{3}$$