ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left|\sin\frac{A-B}{2}.\sin\frac{B-C}{2}.\sin\frac{C-A}{2}\right| \le \frac{1}{24\sqrt{3}}.\frac{\sqrt{(s^2-12Rr-3r^2)^3}}{R^2r}$$

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$$\sin^2\frac{A-B}{2}.\sin^2\frac{B-C}{2}.\sin^2\frac{C-A}{2} = \prod_{cyc}\frac{\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}}{\sin^2\frac{A}{2}} = \prod_{cyc}\frac{\frac{(b-c)^2}{16R^2}}{\sin^2\frac{A}{2}}$$

$$= \frac{1}{256.16R^6}.\frac{16R^2}{r^2}.\prod_{cyc}(b-c)^2 \therefore \prod_{cyc}\sin^2\frac{B-C}{2} = \frac{1}{256R^4r^2}.\prod_{cyc}(b-c)^2 \to (1)$$

$$\text{Now}, \prod_{cyc}(b-c)^2 = \sum_{cyc}a^4b^2 + \sum_{cyc}a^2b^4 - 2abc\sum_{cyc}a^3 - 2\sum_{cyc}a^3b^3$$

$$+2abc\left(\sum_{cyc}a^2b + \sum_{cyc}ab^2\right) - 6a^2b^2c^2$$

$$= \sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2 - c^2\right)\right) - 16Rrs^2(s^2-6Rr-3r^2)$$

$$-2\left(\left(\sum_{cyc}ab\right)^3 - 3abc\prod_{cyc}(b+c)\right) + 2abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) - 3abc\right) - 6a^2b^2c^2$$

$$= 2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 16Rrs^2(s^2-6Rr-3r^2)$$

$$-2\left((s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)\right) + 16Rrs^2(s^2+4Rr+r^2)$$

$$-240R^2r^2s^2 = 4r^2(-s^4+(4R^2+20Rr-2r^2)s^2 - r(4R+r)^3)$$

$$\therefore \prod_{cyc}\sin^2\frac{B-C}{2} \overset{\text{via }(1)}{=} \frac{1}{256R^4r^2}.4r^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3)$$

$$\overset{?}{\le} \frac{(s^2-12Rr-3r^2)^3}{3.576R^4r^2} \Leftrightarrow s^4 - (36Rr-18r^2)s^2 + 81r^2(2R-r)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow s^4 - 2s^2.9r(2R-r) + \left(9r(2R-r)\right)^2 \overset{?}{\ge} 0 \Leftrightarrow \left(s^2-9r(2R-r)\right)^2 \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore \prod_{cyc}\sin^2\frac{B-C}{2} \le \frac{(s^2-12Rr-3r^2)^3}{3.576R^4r^2}$$

$$\Rightarrow \left|\sin\frac{A-B}{2}.\sin\frac{B-C}{2}.\sin\frac{C-A}{2}\right| \le \frac{1}{24\sqrt{3}}.\frac{\sqrt{(s^2-12Rr-3r^2)^3}}{R^2r}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)