ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$:\frac{3}{4} \le \cos^2 A + \cos^2 B + \cos^2 C \le \frac{33}{8} - \frac{s^2}{2R^2}$$

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Solution by Tapas Das-India

$$\cos^2 A + \cos^2 B + \cos^2 C = 3 - \sum sin^2 A = 3 - \frac{\sum a^2}{4R^2} \ (1)$$

$$From\ (1) \cos^2 A + \cos^2 B + \cos^2 C = 3 - \frac{\sum a^2}{4R^2} \overset{Leibniz}{\ge} 3 - \frac{9R^2}{4R^2} = 3 - \frac{9}{4} = \frac{3}{4}$$

$$From\ (1) \cos^2 A + \cos^2 B + \cos^2 C = 3 - \frac{\sum a^2}{4R^2} = 3 - \frac{2(s^2 - r^2 - 4Rr)}{4R^2} =$$

$$= 3 + \frac{r^2 + 4Rr}{2R^2} - \frac{s^2}{2R^2} = 3 + \frac{1}{2}\left(\frac{r}{R}\right)^2 + 2\frac{r}{R} - \frac{s^2}{2R^2} \overset{Euler}{\le} 3 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + 2.\frac{1}{2} - \frac{s^2}{2R^2} =$$

$$= 4 + \frac{1}{8} - \frac{s^2}{2R^2} = \frac{33}{8} - \frac{s^2}{2R^2}$$

Equality holds for an equilateral triangle