ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$-\frac{3(R-2r)}{R}\leq(2\cos A-1)(2\cos B-1)(2\cos C-1)\leq\frac{(R-2r)(16R^2-Rr-8r^2)}{(16R-5r)R^2}$$

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Solution by Tapas Das-India

$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=$$
$$=-1+2\sum\cos A-4\sum\cos A\cos B+8\cos A\cos B\cos C=$$
$$=-1+2\left(1+\frac{r}{R}\right)-4\frac{s^2+r^2-4R^2}{4R^2}+8\,\frac{s^2-(2R+r)^2}{4R^2}=$$
$$=\frac{-R^2+2R^2+2Rr-s^2-r^2+4R^2+2s^2-2(4R^2+4Rr+r^2)}{R^2}=$$
$$=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\ (1)$$
$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\overset{Gerretsen}{\geq}$$
$$\geq\frac{16Rr-5r^2-3R^2-6Rr-3r^2}{R^2}=\frac{-3R^2+10Rr-8r^2}{R^2}\overset{Euler}{\geq}$$
$$\geq\frac{-3R^2+10Rr-8r.\frac{R}{2}}{R^2}=\frac{-3R^2+6Rr}{R^2}=-\frac{3(R-2r)}{R}$$

$$(2\cos A-1)(2\cos B-1)(2\cos C-1)=$$
$$=\frac{s^2-3R^2-6Rr-3r^2}{R^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-3R^2-6Rr-3r^2}{R^2}=$$
$$=\frac{R(R-2r)}{R^2}=\frac{(R-2r)R(16R-5r)}{(16R-5r)R^2}=\frac{(R-2r)(16R^2-5Rr)}{(16R-5r)R^2}=$$
$$=\frac{(R-2r)(16R^2-Rr-4Rr)}{(16R-5r)R^2}\overset{Euler}{\leq}\frac{(R-2r)(16R^2-Rr-4.2r.r)}{(16R-5r)R^2}=$$
$$=\frac{(R-2r)(16R^2-Rr-8r^2)}{(16R-5r)R^2}$$

$$Equality\ holds\ for\ A=B=C$$