ROMANIAN MATHEMATICAL MAGAZINE

Let ABC be the triangle in which the following relationship holds:

$$\left(1-\frac{r_a}{r_b}\right)\left(1-\frac{r_a}{r_c}\right)=2$$

$$\textit{Prove that}:\ \frac{1}{e^{\sin B}}+\frac{1}{e^{\sin C}}\geq\frac{2}{e^{\frac{1}{\sqrt{2}}}}$$

Proposed by Tapas Das-India

Solution by Mirsadix Muzefferov-Azerbaijan

$$\left(1-\frac{r_a}{r_b}\right)\left(1-\frac{r_a}{r_c}\right)=2\ =>$$

$$(r_b-r_a)(r_c-r_a)=2.r_b.r_c\ =>\ r_br_c-r_ar_b-r_ar_c+(r_a)^2=2.r_b.r_c$$

$$r_ar_b+r_br_c+r_ar_c=(r_a)^2 => \boxed{r_a=p.\ tg\frac{A}{2}}$$

$$tg\frac{A}{2}.tg\frac{B}{2}+tg\frac{B}{2}.tg\frac{C}{2}+tg\frac{A}{2}.tg\frac{C}{2}=tg^2\frac{A}{2}\rightarrow tg^2\frac{A}{2}=1\rightarrow$$

$$A=90^\circ\ (\textit{Right triangle})$$

$$\textit{Here}:\ \boxed{tg\frac{A}{2}.tg\frac{B}{2}+tg\frac{B}{2}.tg\frac{C}{2}+tg\frac{A}{2}.tg\frac{C}{2}=1}\ (\textit{true})$$

$$\frac{1}{e^{\sin B}}+\frac{1}{e^{\sin C}}=e^{-\sin B}+e^{-\sin C}\overset{A-G}{\geq}2\sqrt{e^{-\sin B-\sin C}}$$

$$\sin B+\sin C=2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}=2\cos\frac{A}{2}.\cos\frac{B-C}{2}\overset{A=90^\circ}{=}2.\frac{1}{\sqrt{2}}.\cos\frac{B-C}{2}\leq\frac{2}{\sqrt{2}}$$

$$\frac{1}{e^{\sin B}}+\frac{1}{e^{\sin C}}\geq2.\sqrt{e^{-\frac{2}{\sqrt{2}}}}=\frac{2}{e^{\frac{1}{\sqrt{2}}}}\quad(\textit{Proved})$$