ROMANIAN MATHEMATICAL MAGAZINE

Let ABC be a triangle with the measures of all its angle smaller than $\frac{2\pi}{3}$ and T its Torricelli's point. Prove that :

$$m_a^2.TA + m_b^2.TB + m_c^2.TC \ge \sqrt[4]{432F^6}$$

Proposed by Tapas Das-India

Solution 1 by Mohamed Amine Ben Ajiba-Morocco

Since m_a, m_b, m_c can be the sides of a triangle Δ_m with area $F_m = \frac{3}{4}F$, then by using Oppenheim's inequality in triangle Δ_m, we have, for all $x, y, z > 0$,

$$m_a^2.x + m_b^2.y + m_c^2.z \ge 4F_m\sqrt{xy + yz + zx}. \qquad (1)$$

Let $x = TA, y = TB, z = TC$. Since we have

$$TA.TB + TB.TC + TC.TA = \frac{4}{\sqrt{3}}.\left(\frac{1}{2}TA.TB.\sin\frac{2\pi}{3} + \frac{1}{2}TB.TC.\sin\frac{2\pi}{3} + \frac{1}{2}TC.TA.\sin\frac{2\pi}{3}\right)$$

$$= \frac{4\sqrt{3}}{3}([TAB] + [TBC] + [TCA]) = \frac{4\sqrt{3}}{3}F,$$

then the inequality (1) becomes $m_a^2.TA + m_b^2.TB + m_c^2.TC \ge 4.\frac{3}{4}F.\sqrt{\frac{4\sqrt{3}}{3}F} = \sqrt[4]{432F^6}$.

Equality holds iff ΔABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

Let $x = TA, y = TB, z = TC$. We have

$$F = [TAB] + [TBC] + [TCA] = \frac{1}{2}TA.TB.\sin\frac{2\pi}{3} + \frac{1}{2}TB.TC.\sin\frac{2\pi}{3} + \frac{1}{2}TC.TA.\sin\frac{2\pi}{3}$$

$$= \frac{\sqrt{3}}{4}(TA.TB + TB.TC + TC.TA) = \frac{\sqrt{3}}{4}(xy + yz + zx),$$

$$a^2 = TB^2 + TC^2 - 2TB.TC.\cos\frac{2\pi}{3} = y^2 + z^2 + yz \text{ (and analogs)}.$$

$$4m_a^2 = 2(b^2 + c^2) - a^2 = 4x^2 + y^2 + z^2 + 2xy + 2xz - yz \text{ (and analogs)}.$$

$$4(m_a^2.TA + m_b^2.TB + m_c^2.TC) = \sum_{cyc} 4m_a^2.TA = \sum_{cyc}(4x^2 + y^2 + z^2 + 2xy + 2xz - yz)x$$

$$= 4\sum_{cyc} x^3 + 3\sum_{cyc} x^2(y+z) - 3xyz \overset{AM-GM}{\ge} 3\sum_{cyc} x^3 + 3\sum_{cyc} x^2(y+z)$$

$$= 3\sum_{cyc} x^2.\sum_{cyc} x \ge 3\sum_{cyc} yz.\sqrt{3\sum_{cyc} yz} = \sqrt{3(xy+yz+zx)}^3 = 4\sqrt[4]{432F^6}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.