ROMANIAN MATHEMATICAL MAGAZINE

If in Δ ABC holds : $\cos A + 2\cos B + \cos C = 2$, then :

$$9 \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 6 + \frac{3R}{2r}$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$\cos A + 2\cos B + \cos C = 2 \Rightarrow 2\cos B + \left(1 + \frac{r}{R} - \cos B\right) = 2$$
$$\Rightarrow \cos B = 1 - \frac{r}{R} \rightarrow (1)$$

$$\text{Again}, \cos A + 2\cos B + \cos C = 2 \Rightarrow \cos A + \cos C = 2(1 - \cos B)$$

$$\Rightarrow 2\sin\frac{B}{2}\cos\frac{C-A}{2} = 4\sin^2\frac{B}{2} \Rightarrow \frac{c+a}{b}\sin\frac{B}{2} = 2\sin\frac{B}{2} \Rightarrow c + a + b = 3b$$

$$\Rightarrow 2s = 6R\sin B \Rightarrow s^2 = 9R^2(1-\cos^2 B) \overset{\text{via }(1)}{=} 9R^2\left(1 - \left(1 - \frac{r}{R}\right)^2\right)$$
$$= 9R^2\left(\frac{2r}{R} - \frac{r^2}{R^2}\right) = 9R^2\left(\frac{2Rr - r^2}{R^2}\right) \Rightarrow s^2 = 18Rr - 9r^2 \rightarrow (2)$$

$$\text{Now}, (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 6 + \frac{3R}{2r} \Leftrightarrow \frac{2s(s^2+4Rr+r^2)}{4Rrs} \leq \frac{3R+12r}{2r}$$

$$\Leftrightarrow s^2 \leq 3R^2 + 8Rr - r^2 \overset{\text{via }(2)}{\Leftrightarrow} 18Rr - 9r^2 \leq 3R^2 + 8Rr - r^2$$

$$\Leftrightarrow 3R^2 - 10Rr + 8r^2 \geq 0 \Leftrightarrow (R-2r)(3(R-2r)+2r) \geq 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 6 + \frac{3R}{2r} \text{ and finally}, (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{\text{AM-HM}}{\geq} 9$$

$$\therefore 9 \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 6 + \frac{3R}{2r}\ \forall\, \Delta \text{ ABC with } \cos A + 2\cos B + \cos C = 2$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$