ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC, the following relationship holds :

$$\sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2} \le \frac{3}{2}max(m_a, m_b, m_c)$$

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$$m_a \overset{?}{\le} \sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2} \Leftrightarrow 8m_a^2 \overset{?}{\le} (b^2+c^2).\left(4\cos^2\frac{A}{2}\right)$$

$$\Leftrightarrow 2(2b^2+2c^2-a^2)bc \overset{?}{\le} (b^2+c^2)((b+c)^2-a^2)$$

$$\Leftrightarrow 4bc(b^2+c^2)-2a^2bc \overset{?}{\le} (b^2+c^2)^2+2bc(b^2+c^2)-a^2(b^2+c^2)$$

$$\Leftrightarrow \left((b^2+c^2)^2-a^2(b^2+c^2)\right)-(2bc(b^2+c^2)-2a^2bc) \overset{?}{\ge} 0$$

$$\Leftrightarrow (b^2+c^2)(b^2+c^2-a^2)-2bc(b^2+c^2-a^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow (b^2+c^2-a^2)(b^2+c^2-2bc) \overset{?}{\ge} 0 \Leftrightarrow (b^2+c^2-a^2)(b-c)^2 \overset{?}{\ge} 0 \to \text{true}$$

$$\because \Delta\text{ ABC being acute} \Rightarrow b^2+c^2-a^2 > 0 \therefore m_a \le \sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} m_a \sin\frac{A}{2} \overset{CBS}{\le} \sqrt{3}.\sqrt{\sum_{cyc} m_a^2 \sin^2\frac{A}{2}} \le \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{b^2+c^2}{2}.\cos^2\frac{A}{2}\sin^2\frac{A}{2}\right)}$$

$$= \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{b^2+c^2}{2}.\frac{a^2}{16R^2}\right)} = \frac{\sqrt{3}}{2}.\sqrt{\sum_{cyc}\frac{b^2c^2}{4R^2}} = \frac{\sqrt{3}}{2}.\sqrt{\sum_{cyc} h_a^2}$$

$$\therefore \sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}.\sqrt{3\sum_{cyc} h_a^2} \text{ and again, } \frac{3}{2}max(m_a, m_b, m_c) \ge \frac{1}{2}\sum_{cyc} m_a \overset{Tereshin}{\ge}$$

$$\frac{1}{2}\sum_{cyc}\frac{b^2+c^2}{4R} = \frac{1}{4R}.\sum_{cyc} a^2 \ge \frac{1}{4R}.\sqrt{3\sum_{cyc} b^2c^2} = \frac{1}{2}\sqrt{3\sum_{cyc}\frac{b^2c^2}{4R^2}} = \frac{1}{2}.\sqrt{3\sum_{cyc} h_a^2}$$

$$\therefore \frac{1}{2}.\sqrt{3\sum_{cyc} h_a^2} \le \frac{3}{2}max(m_a, m_b, m_c) \text{ and so, } \sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}.\sqrt{3\sum_{cyc} h_a^2}$$

$$\le \frac{3}{2}max(m_a, m_b, m_c)\ \forall \text{ acute } \Delta\text{ ABC}, '' ='' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$