In any Δ ABC, the following relationship holds :

$$\frac{6\sum_{cyc} m_a^2}{\sqrt{a^2+b^2+c^2}} \ge \sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right) \ge \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right)-8\left(\frac{r}{R}\right)^2}$$

Proposed by Tapas Das-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}(s_b+s_c)\sin^2 A+\sum_{cyc}(m_b+m_c)\sin^2 A=\sum_{cyc}(\sin^2 B+\sin^2 C)s_a+\sum_{cyc}(\sin^2 B+\sin^2 C)m_a =$$

$$=\sum_{cyc}\frac{b^2+c^2}{4R^2}\cdot\frac{2bc}{b^2+c^2}m_a+\sum_{cyc}\frac{b^2+c^2}{4R^2}.m_a=\frac{1}{4R^2}\sum_{cyc}(b+c)^2 m_a \le \frac{1}{4R^2}\sum_{cyc}2(b^2+c^2)m_a$$

$$=\frac{2}{R}\sum_{cyc}\frac{b^2+c^2}{4R}.m_a \overset{Tereshin}{\le} \frac{2}{R}\sum_{cyc}{m_a}^2 \overset{Leibniz}{\le} \frac{6({m_a}^2+{m_b}^2+{m_c}^2)}{\sqrt{a^2+b^2+c^2}}$$

Again, we have

$$\sum_{cyc}(s_b+s_c)\sin^2 A+\sum_{cyc}(m_b+m_c)\sin^2 A=\frac{1}{4R^2}\sum_{cyc}(b+c)^2 m_a \ge$$

$$\ge \frac{1}{4R^2}\sum_{cyc}(a+2(s-a))(b+c)w_a \overset{AM-GM}{\ge} \frac{1}{4R^2}\sum_{cyc}2\sqrt{2a(s-a)}.2\sqrt{bcs(s-a)}=$$

$$=\frac{\sqrt{2sabc}}{R^2}\sum_{cyc}(s-a)=\frac{2s\sqrt{2s^2Rr}}{R^2} \overset{Gerresten}{\ge} \frac{2s\sqrt{2(16Rr-5r^2)Rr}}{R^2}=\frac{2F}{R}\sqrt{2\left(16-\frac{5r}{R}\right)}=$$

$$=\frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2+\frac{4}{9}\left(1-\frac{2r}{R}\right)\left(8-\frac{9r}{R}\right)} \overset{Euler}{\ge} \frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_b \overset{?}{\le} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right) \overset{?}{\le} \frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2 \overset{?}{\ge} 0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\ge} 0$$

$$\to \text{true} \Rightarrow m_a m_b \le \frac{2c^2+ab}{4} \text{ and analogs} \to (1)$$

$$\text{Now, } \sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\le$$

$$\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\left(\because s_a=\frac{2bc}{b^2+c^2}.m_a\overset{A-G}{\le}m_a \text{ and analogs}\right)$$

$$=\frac{2}{4R^2}\sum_{cyc}a^2(m_b+m_c)\overset{CBS}{\le}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b+m_c)^2}$$

$$\overset{via\,(1)}{\le}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b^2+m_c^2)+\sum_{cyc}\left(2a^2.\frac{2a^2+bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}\frac{a^2(b^2+c^2+4a^2)}{4}+\sum_{cyc}\left(\frac{4a^4+2a^2bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{1}{2}\left(4\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+\sum_{cyc}a^2b^2+8Rrs^2\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{9((s^2+4Rr+r^2)^2-16Rrs^2)-64r^2s^2+8Rrs^2}{2}}\overset{?}{\le}\frac{6\sum_{cyc}m_a^2}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{9\sum_{cyc}a^2}{2\sqrt{\sum_{cyc}a^2}}\Leftrightarrow 9\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-64r^2s^2+8Rrs^2\overset{?}{\le}162R^4$$

$$\Leftrightarrow 9s^4-(64Rr+46r^2)s^2+9r^2(4R+r)^2-162R^4\boxed{\underset{(*)}{\overset{?}{\le}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$ $\therefore$ in order to prove $(*)$, it suffices to prove :

$$9s^4-(64Rr+46r^2)s^2+9r^2(4R+r)^2-162R^4$$
$$\le 9s^4-9s^2(4R^2+20Rr-2r^2)+9r(4R+r)^3\Leftrightarrow$$

$$(18R^2+58Rr-32r^2)s^2\boxed{\overset{(**)}{\le}}81R^4+288R^3r+144R^2r^2+18Rr^3 \text{ and}$$

again, LHS of $(**)\overset{Gerretsen}{\le}(18R^2+58Rr-32r^2)(4R^2+4Rr+3r^2)\overset{?}{\le}$

$$\text{RHS of }(**)\Leftrightarrow 9t^4-16t^3-14t^2-28t+96\ge 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(9t^2+20t+30)+12\right)\ge 0\to \text{true}\because t\overset{Euler}{\ge}2$$

$$\Rightarrow (**)\Rightarrow (*)\text{ is true}$$

$$\therefore\sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\le\frac{6\sum_{cyc}m_a^2}{\sqrt{a^2+b^2+c^2}}$$

$$\text{Again, }\sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\ge 2\sum_{cyc}\left((h_b+h_c)\sin^2 A\right)$$

$$= \frac{1}{4R^3} \cdot \sum_{cyc} a^2(ca + ab) = \frac{1}{4R^3} \cdot \sum_{cyc} \left(a^2 \left(\sum_{cyc} ab - bc \right) \right)$$

$$= \frac{(s^2 + 4Rr + r^2)(s^2 - 4Rr - r^2) - 4Rrs^2}{2R^3} \overset{?}{\geq} \frac{6F}{R} \cdot \sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

$$= \frac{6rs}{R^2} \cdot \sqrt{10Rr - 8r^2}$$

$$\Leftrightarrow \frac{\left((s^2 + 4Rr + r^2)(s^2 - 4Rr - r^2) - 4Rrs^2\right)^2}{4R^6} \overset{?}{\geq} \frac{36r^2s^2(10Rr - 8r^2)}{R^4}$$

$$\Leftrightarrow s^8 - 8Rrs^6 - r^2(16R^2 + 16Rr + 2r^2)s^4 - Rr^3(1312R^2 - 1216Rr - 8r^2)s^2$$

$$+r^4(4R + r)^4 \boxed{\underset{(***)}{\overset{?}{\geq}}} 0 \text{ and } \because (s^2 - 16Rr + 5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

to prove $(***)$, it suffices to prove : LHS of $(***) \geq (s^2 - 16Rr + 5r^2)^4$

$$\Leftrightarrow (14R - 5r)s^6 - r(388R^2 - 236Rr + 38r^2)s^4$$

$$+r^2(3768R^3 - 3536R^2r + 1202Rr^2 - 125r^3)$$

$$-r^3(16320R^4 - 20544R^3r + 9576R^2r^2 - 2004Rr^3 + 156r^4) \boxed{\overset{(****)}{\geq}} 0 \text{ and}$$

$\because (14R - 5r)(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(****)$, it suffices to prove : LHS of $(****) \geq (14R - 5r)(s^2 - 16Rr + 5r^2)^3$

$$\Leftrightarrow (284R^2 - 214Rr + 37r^2)s^4 - r(6984R^3 - 7024R^2r + 2248Rr^2 - 250r^3)$$

$$+r^2(41024R^4 - 53696R^3r + 26424R^2r^2 - 5746Rr^3 + 469r^4) \boxed{\overset{(*****)}{\geq}} 0 \text{ and}$$

$\because (284R^2 - 214Rr + 37r^2)(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(****)$, it suffices to prove : LHS of $(*****) \geq$

$$(284R^2 - 214Rr + 37r^2)(s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (526R^3 - 666R^2r + 269Rr^2 - 30r^3)s^2 \boxed{\overset{(******)}{\geq}}$$

$$r(7920R^4 - 11632R^3r + 6097R^2r^2 - 1381Rr^3 + 114r^4)$$

Finally, $(526R^3 - 666R^2r + 269Rr^2 - 30r^3)s^2 \overset{\text{Gerretsen}}{\geq}$

$$(526R^3 - 666R^2r + 269Rr^2 - 30r^3)(16Rr - 5r^2) \overset{?}{\geq}$$

$$r(7920R^4 - 11632R^3r + 6097R^2r^2 - 1381Rr^3 + 114r^4)$$

$$\Leftrightarrow 496t^4 - 1654t^3 + 1537t^2 - 444t + 36 \geq 0$$

$$\Leftrightarrow (t - 2)\left((t - 2)(496t^2 + 330t + 873) + 1728\right) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (******) \Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \text{ is true}$$

$$\therefore \sum_{cyc} \left((s_b + s_c)\sin^2 A\right) + \sum_{cyc} \left((m_b + m_c)\sin^2 A\right) \geq \frac{6F}{R} \cdot \sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

and so, $\dfrac{6\sum_{cyc} m_a^2}{\sqrt{a^2 + b^2 + c^2}} \geq \sum_{cyc} \left((s_b + s_c)\sin^2 A\right) + \sum_{cyc} \left((m_b + m_c)\sin^2 A\right)$

$\geq \dfrac{6F}{R} \cdot \sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$ $\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)