ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}}\le\frac{1}{8}\left(1+\frac{5R}{2r}\right)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z \therefore s=x+y+z$$

$$\Rightarrow a=y+z, b=z+x \text{ and } c=x+y$$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{F^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\Rightarrow\frac{s^2}{r^2}\overset{(*)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\text{ and }1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(**)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Also, }\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{y+z}{z+x}\Rightarrow\sum\frac{a}{b}\overset{(***)}{=}\frac{\sum_{cyc}\left((x+y)(y+z)^2\right)}{\prod_{cyc}(y+z)}\therefore(*),(**),(***)\Rightarrow$$

$$\frac{s^2}{r^2}\ge\left(\sum_{cyc}\frac{a}{b}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum_{cyc}x\right)^3}{xyz}\ge$$

$$\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}\left((x+y)(y+z)^2\right)}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow\left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc}x\right)^3\ge\left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}\left((x+y)(y+z)^2\right)\right)$$

$$\Leftrightarrow\sum_{cyc}x^2y^4+\sum_{cyc}x^3y^3\ \boxed{\overset{(\bullet)}{\ge}}\ xyz\sum_{cyc}x^2y+3x^2y^2z^2$$

$$\text{Now, if } u,v,w>0, \textit{then}: v^3+v^3+u^3\overset{A-G}{\ge}3v^2u,$$

$$w^3+w^3+v^3\overset{A-G}{\ge}3w^2v\textit{ and }u^3+u^3+w^3\overset{A-G}{\ge}3u^2w\textit{ and adding these three}:$$

$$\sum_{cyc}u^3\ge\sum_{cyc}uv^2\textit{ and choosing }u=xy, v=yz\text{ and }w=zx,\text{ we get}:$$

$$\sum_{cyc} x^3y^3 \overset{(\bullet\bullet)}{\geq} xyz\left(\sum_{cyc} x^2y\right) \text{ and } \sum_{cyc} x^2y^4 \underbrace{\overset{A-G}{\geq}}_{(\bullet\bullet\bullet)} 3x^2y^2z^2 \therefore (\bullet\bullet) + (\bullet\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \boxed{\frac{s^2}{r^2} \geq \left(\sum_{cyc} \frac{a}{b}\right)\left(1 + \frac{4R}{r}\right)} \to (1)$$

$$\text{Moreover, } \sum_{cyc} \frac{b}{a} = \sum_{cyc} \frac{z+x}{y+z} \Rightarrow \sum_{cyc} \frac{b}{a} \overset{(****)}{=} \frac{\sum_{cyc}((x+y)^2(y+z))}{\prod_{cyc}(y+z)} \therefore (*), (**), (****)$$

$$\Rightarrow \frac{s^2}{r^2} \geq \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{xyz} \geq$$

$$\left(\frac{xyz + \prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}((x+y)^2(y+z))}{\prod_{cyc}(y+z)}\right)$$

$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc} x\right)^3 \geq \left(xyz + \prod_{cyc}(y+z)\right)\left(\sum_{cyc}((x+y)^2(y+z))\right)$$

$$\Leftrightarrow \sum_{cyc} x^4y^2 + \sum_{cyc} x^3y^3 \boxed{\overset{(\bullet\bullet\bullet\bullet)}{\geq}} xyz\sum_{cyc} xy^2 + 3x^2y^2z^2$$

$$\text{Now, if } u, v, w > 0, \textit{then}: u^3 + u^3 + v^3 \overset{A-G}{\geq} 3u^2v,$$

$$v^3 + v^3 + w^3 \overset{A-G}{\geq} 3v^2w \textit{ and } w^3 + w^3 + u^3 \overset{A-G}{\geq} 3w^2u \textit{ and adding these three}:$$

$$\sum_{cyc} u^3 \geq \sum_{cyc} u^2v \textit{ and choosing } u = xy, v = yz \text{ and } w = zx, \text{we get}:$$

$$\sum_{cyc} x^3y^3 \overset{(\bullet\bullet\bullet\bullet\bullet)}{\geq} xyz\left(\sum_{cyc} xy^2\right) \textit{and} \sum_{cyc} x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(\bullet\bullet\bullet\bullet\bullet\bullet)} 3x^2y^2z^2 \therefore (\bullet\bullet\bullet\bullet\bullet) + (\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true} \therefore \boxed{\frac{s^2}{r^2} \geq \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right)} \to (2)$$

$$\text{Now, } m_a m_b \overset{?}{\leq} \frac{2c^2 + ab}{4} \Leftrightarrow \left(\frac{2b^2 + 2c^2 - a^2}{4}\right)\left(\frac{2c^2 + 2a^2 - b^2}{4}\right) \overset{?}{\leq} \frac{(2c^2 + ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2\overset{?}{\geq}0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\geq}0\Leftrightarrow (a-b)^2(a+b+c)(a+b-c)\overset{?}{\geq}0$$

$$\rightarrow \text{true}\Rightarrow \boxed{m_a m_b\leq\frac{2c^2+ab}{4}}\ \textit{and analogs}\rightarrow(3)$$

$$\text{Now, }\frac{1}{2}\left(1+\frac{5m_am_bm_c}{9F^2}\left(\sum_{cyc}m_a\right)\right)\overset{\substack{\text{Lascu}\\ \text{and}\\ \text{Tereshin}}}{\geq}$$

$$\frac{1}{2}\left(1+\frac{5}{9r^2s^2}\left(\prod_{cyc}\left(\frac{b+c}{2}\cos\frac{A}{2}\right)\right)\left(\sum_{cyc}\frac{b^2+c^2}{4R}\right)\right)$$

$$=\frac{1}{2}\left(1+\frac{5}{9r^2s^2}\left(\frac{2s(s^2+2Rr+r^2)}{8}\cdot\frac{s}{4R}\right)\left(\frac{s^2-4Rr-r^2}{R}\right)\right)$$

$$\overset{?}{\geq}\frac{s^2}{r(4R+r)}\Leftrightarrow\frac{5(s^2+2Rr+r^2)(s^2-4Rr-r^2)}{144R^2r^2}\overset{?}{\geq}\frac{2s^2-4Rr-r^2}{r(4R+r)}$$

$$\Leftrightarrow(20R+5r)s^4-rs^2(328R^2+10Rr)+r^2(416R^3-16R^2r-50Rr^2-5r^3)\boxed{\overset{?}{\underset{(\blacksquare)}{\geq}}}0$$

$$\textit{and}\because(20R+5r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(\blacksquare),$$

$$\text{it suffices to prove : LHS of }(\blacksquare)\geq(20R+5r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(156R^2-25Rr-25r^2)s^2\boxed{\overset{(\blacksquare\blacksquare)}{\geq}}r(2352R^3-952R^2r-125Rr^2+65r^3)$$

$$\textit{Finally},(156R^2-25Rr-25r^2)s^2\overset{\text{Gerretsen}}{\geq}(156R^2-25Rr-25r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq}r(2352R^3-952R^2r-125Rr^2+65r^3)\Leftrightarrow 24t^3-38t^2-25t-10\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(24t^2+10t-5)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true}$$

$$\therefore\frac{1}{2}\left(1+\frac{5m_am_bm_c}{9F^2}\left(\sum_{cyc}m_a\right)\right)\geq\frac{s^2}{r(4R+r)}\geq\sqrt{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{b}{a}\right)}$$

$$\left(\begin{array}{c}\because (1) \text{ and } (2) \Rightarrow \dfrac{s^2}{r(4R+r)} \geq \left(\sum_{cyc}\dfrac{a}{b}\right), \left(\sum_{cyc}\dfrac{b}{a}\right) \\ \Rightarrow \sqrt{\left(\sum_{cyc}\dfrac{a}{b}\right)\left(\sum_{cyc}\dfrac{b}{a}\right)} \leq \dfrac{s^2}{r(4R+r)}\end{array}\right)$$

$$\therefore \boxed{\frac{1}{2}\left(1+\frac{5m_a m_b m_c}{9F^2}\left(\sum_{cyc} m_a\right)\right) \geq \sqrt{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{b}{a}\right)}} \to (4)$$

Now, implementing (4) on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians and area as a consequence of trivial calculations are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ and $\frac{F}{3}$ respectively, we get :

$$\frac{1}{2}\left(1+\frac{5.\frac{abc}{8}}{9.\frac{F^2}{9}}\left(\frac{1}{2}\sum_{cyc} a\right)\right) \geq \sqrt{\left(\sum_{cyc}\frac{\frac{2m_a}{3}}{\frac{2m_b}{3}}\right)\left(\sum_{cyc}\frac{\frac{2m_b}{3}}{\frac{2m_a}{3}}\right)}$$

$$\Rightarrow \sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)} \leq \frac{1}{2}\left(1+\frac{5.\frac{4Rrs}{8}}{9.\frac{r^2s^2}{9}}.s\right)$$

$$\therefore \boxed{\sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)} \leq \frac{1}{2}\left(1+\frac{5R}{2r}\right)} \to (5)$$

$$\text{Now, } \frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}} \overset{A-G}{\leq} \sum_{cyc}\frac{m_a}{\sqrt{4bc+8a^2}}$$

$$\overset{\text{via }(3)}{\leq} \sum_{cyc}\frac{m_a}{\sqrt{16m_b m_c}} = \frac{1}{4}\sum_{cyc}\left(\sqrt{\frac{m_a}{m_b}}.\sqrt{\frac{m_a}{m_c}}\right) \overset{A-G}{\leq} \frac{1}{4}.\sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)}$$

$$\overset{\text{via }(5)}{\leq} \frac{1}{8}\left(1+\frac{5R}{2r}\right) \therefore \frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}}$$

$$\leq \frac{1}{8}\left(1+\frac{5R}{2r}\right)\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(b+c)^2+8a^2=(b+c)^2+4a^2+4a^2\geq 4a(b+c)+4a^2=$$

$$=4a\cdot 2s=8sa\quad \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{m_a}{\sqrt{(b+c)^2+8a^2}}\leq \sum_{cyc}\frac{m_a}{\sqrt{8sa}}\overset{CBS}{\leq}\sqrt{\frac{3}{8s}\sum_{cyc}\frac{m_a{}^2}{a}}=\sqrt{\frac{3}{8s}\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4a}}$$

$$=\sqrt{\frac{3}{32sabc}\left(2\sum_{cyc}bc(b^2+c^2)-abc\sum_{cyc}a\right)}$$

$$=\sqrt{\frac{3}{128s^2Rr}\left[2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}bc\right)-3abc\sum_{cyc}a\right]}$$

$$=\sqrt{\frac{3}{128s^2Rr}[4(s^2-r^2-4Rr)(s^2+r^2+4Rr)-3.4Rsr2s]}$$

$$=\sqrt{\frac{3[s^4-r^2(4R+r)^2-6s^2Rr]}{32s^2Rr}}$$

$$=\sqrt{\frac{3}{32Rr}\left[s^2-6Rr-\frac{r^2(4R+r)^2}{s^2}\right]}\overset{\substack{Gerretsen\\Doucet}}{\leq}\sqrt{\frac{3}{32Rr}\left[4R^2-2Rr+3r^2-\frac{r^2.3s^2}{s^2}\right]}$$

$$=\sqrt{\frac{3(2R-r)}{16r}}\overset{AM-GM}{\leq}\frac{1}{2}\left(\frac{3}{4}+\frac{2R-r}{4r}\right)=\frac{1}{4}\left(1+\frac{R}{r}\right)\overset{Euler}{\leq}\frac{1}{8}\left(1+\frac{5R}{2r}\right).$$

Equality holds iff ΔABC is equilateral.