ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}+\left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}}+\left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}}\ge\left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)-\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right)}$$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2\ge 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\ge\frac{12}{\left(\sum_{cyc}a^2\right)^2}\Leftrightarrow$$

$$\left(\sum_{cyc}a^2\right)^2-3a^2(2b^2+2c^2-a^2)\ge 0\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-3a^2\left(2\sum_{cyc}a^2-3a^2\right)\ge 0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^2-6a^2\sum_{cyc}a^2+9a^4\ge 0\Leftrightarrow\left(\sum_{cyc}a^2-3a^2\right)^2\ge 0$$

$$\Leftrightarrow(b^2+c^2-2a^2)^2\ge 0\to\text{true}\Rightarrow am_a\le\frac{\sum_{cyc}a^2}{2\sqrt{3}}\text{ and analogs}\to(1)$$

$$\text{Now, }\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}=\left(1+\left(\frac{n_a^2}{h_a^2}-1\right)\right)^{\frac{m_a}{h_a}}\overset{\text{Bernoulli}}{\ge}1+\left(\frac{n_a^2}{h_a^2}-1\right).\frac{m_a}{h_a}\ge 1\Rightarrow\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}\ge 1$$

$$\text{and analogs}\therefore\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}+\left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}}+\left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}}\ge 3\to(2)$$

$$\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)=\sum_{cyc}\frac{m_a^2}{h_a^2}+3\sum_{cyc}\frac{m_bm_c}{h_bh_c}\le$$

$$\sum_{cyc}\frac{m_a^2}{h_a^2}+2\sum_{cyc}\frac{m_bm_c}{h_bh_c}+\frac{1}{3}.\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2=\frac{4}{3}.\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2$$

$$\Rightarrow\left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)}\le\frac{2s}{\sqrt{3}r}.\sum_{cyc}\frac{am_a}{2rs}\overset{\text{via }(1)}{\le}$$

$$\frac{1}{\sqrt{3}r^2}.\frac{3}{2\sqrt{3}}\sum_{cyc}a^2\therefore\left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)}\le\frac{s^2-4Rr-r^2}{r^2}$$

$$\to (3) \therefore (2) \text{ and } (3) \Rightarrow \text{it suffices to prove : } 3+\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right) \geq$$

$$\frac{s^2-4Rr-r^2}{r^2} \Leftrightarrow \frac{4R^2+Rr}{r^2} \geq \frac{s^2-4Rr-r^2}{r^2} \Leftrightarrow s^2 \leq 4R^2+5Rr+r^2$$

$$\Leftrightarrow (s^2-4R^2-4Rr-3r^2)-r(R-2r) \leq 0 \to \text{true via Gerretsen and Euler}$$

$$\Rightarrow \text{main inequality is true,}'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $n_a \geq h_a$ (and analogs), then

$$\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}+\left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}}+\left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}} \geq 1+1+1=3. \quad (1)$$

By Panaitopol's inequality, we have $\frac{m_a}{h_a} \leq \frac{R}{2r}$ (and analogs), then

$$\sqrt{\sum_{cyc}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)} \leq \sqrt{\sum_{cyc}\left(\frac{R}{2r}+\frac{R}{2r}\right)\left(\frac{R}{2r}+\frac{R}{2r}\right)}=\frac{\sqrt{3}R}{r}. \quad (2)$$

By Doucet's inequality, we have

$$\sum_{cyc}\cot\frac{A}{2}=\frac{s}{r} \leq \frac{4R+r}{\sqrt{3}r}. \quad (3)$$

From (1), (2) and (3), we get

$$\sum_{cyc}\cot\frac{A}{2}.\sqrt{\sum_{cyc}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)}-\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right)$$

$$\leq \frac{4R+r}{\sqrt{3}r}.\frac{\sqrt{3}R}{r}-\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right)=3 \leq \left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}+\left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}}+\left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}}.$$

Equality holds iff ΔABC is equilateral.