ROMANIAN MATHEMATICAL MAGAZINE

If in Δ ABC, the following relationship holds : $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$, then

prove that : $1\le\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\le\frac{R}{r}-1$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}\Rightarrow 2\sin A\sin(B-C)=2\sin C\sin(A-B)$$
$$\Rightarrow\cos(A-B+C)-\cos(A+B-C)=\cos(C-A+B)-\cos(C+A-B)$$
$$\Rightarrow 2\cos(\pi-2B)=\cos(\pi-2A)+\cos(\pi-2C)\Rightarrow -2\cos 2B=-\cos 2A-\cos 2C$$
$$\Rightarrow 1-2\sin^2 A+1-2\sin^2 C=2-4\sin^2 B\Rightarrow\frac{2b^2}{4R^2}=\frac{a^2+c^2}{4R^2}$$
$$\Rightarrow 2b^2=a^2+c^2\to(1)$$
$$\text{Now, }\frac{R}{r}-1-\left(\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\right)\overset{Bandila}{\ge}$$
$$\frac{c}{a}+\frac{a}{c}-1-\left(\frac{a^2}{b^2+c^2}+\frac{c^2}{a^2+b^2}\right)\overset{\text{via }(1)}{=}$$
$$\frac{c^2+a^2-ca}{ca}-\left(\frac{a^2}{\frac{a^2+c^2}{2}+c^2}+\frac{c^2}{a^2+\frac{a^2+c^2}{2}}\right)\overset{?}{\ge}0$$
$$\Leftrightarrow\frac{c^2+a^2-ca}{ca}\overset{?}{\ge}\frac{2a^2(3a^2+c^2)+2c^2(3c^2+a^2)}{(3c^2+a^2)(3a^2+c^2)}$$
$$\Leftrightarrow 3t^6-9t^5+13t^4-14t^3+13t^2-9t+3\overset{?}{\ge}0\ \left(t=\frac{a}{c}\right)$$
$$\Leftrightarrow\frac{1}{16}(t-1)^2\left((12t^2+13)(2t-1)^2+4t+35\right)\overset{?}{\ge}0\to\text{true}$$
$$\therefore\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\le\frac{R}{r}-1$$
$$\text{Again, }\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}-1=\frac{a^2}{\frac{a^2+c^2}{2}+c^2}+\frac{c^2}{a^2+\frac{a^2+c^2}{2}}-1$$
$$=\frac{2a^2(3a^2+c^2)+2c^2(3c^2+a^2)-(3c^2+a^2)(3a^2+c^2)}{(3c^2+a^2)(3a^2+c^2)}$$
$$=\frac{3(a^4-2a^2c^2+c^4)}{(3c^2+a^2)(3a^2+c^2)}=\frac{3(c^2-a^2)^2}{(3c^2+a^2)(3a^2+c^2)}\ge 0$$
$$\therefore\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\ge 1\text{ and so,}$$
$$1\le\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}\le\frac{R}{r}-1\text{ whenever}$$
$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}\Leftrightarrow \sin A\sin(B-C)=\sin C\sin(A-B)$$

$$\Leftrightarrow \cos(A-B+C)-\cos(A+B-C)=\cos(C-A+B)-\cos(C+A-B)$$

$$\overset{A+B+C=\pi}{\overbrace{\Leftrightarrow}}\quad -\cos(2B)+\cos(2C)=-\cos(2A)+\cos(2B)$$

$$\overset{\cos 2x=1-2\sin^2 x}{\overbrace{\Leftrightarrow}}\quad (1-2\sin^2 A)+(1-2\sin^2 C)=2(1-2\sin^2 B)$$

$$\overset{a=2R\sin A}{\overbrace{\Leftrightarrow}}\quad a^2+c^2=2b^2.$$

$$\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}=\frac{a^2}{b^2+c^2}+\frac{c^2}{a^2+b^2}=\frac{2a^2}{a^2+3c^2}+\frac{2c^2}{3a^2+c^2}$$

$$\overset{CBS}{\overbrace{\geq}}\frac{2(a^2+c^2)^2}{a^2(a^2+3c^2)+c^2(3a^2+c^2)}=\frac{2(a^2+c^2)^2}{(a^2+c^2)^2+4a^2c^2}\overset{AM-GM}{\overbrace{\geq}}$$

$$\geq\frac{2(a^2+c^2)^2}{(a^2+c^2)^2+(a^2+c^2)^2}=1.$$

$$\frac{\sin^2 A}{\sin^2 B+\sin^2 C}+\frac{\sin^2 C}{\sin^2 A+\sin^2 B}=\frac{2a^2}{a^2+3c^2}+\frac{2c^2}{3a^2+c^2}=$$

$$=\frac{2a^2}{(a^2+c^2)+2c^2}+\frac{2c^2}{2a^2+(a^2+c^2)}$$

$$\overset{AM-GM}{\overbrace{\leq}}\frac{2a^2}{2ac+2c^2}+\frac{2c^2}{2a^2+2ac}=\frac{a^3+c^3}{ac(a+c)}=\frac{a^2+c^2-ac}{ac}=\frac{a}{c}+\frac{c}{a}-1\overset{Bandila}{\overbrace{\leq}}\frac{R}{r}-1$$

Equality holds iff ΔABC is equilateral.