NEW INEQUALITIES WITH FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper we present new inequalities with Fibonacci and Lucas numbers.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$.
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$.

Application 1. In any triangle ABC with usual notations and the area F holds the following inequality:

$$\frac{a^{F_n}b^{F_{n+2}}}{h_a^{F_{n+1}}} + \frac{b^{F_n}c^{F_{n+2}}}{h_b^{F_{n+1}}} + \frac{c^{F_n}a^{F_{n+2}}}{h_c^{F_{n+1}}} \geq 2^{F_n+F_{n+2}}(\sqrt{3})^{2-F_{n+2}}F^{F_n}$$

Proof.

$$\sum_{cyc}\frac{a^{F_n}b^{F_{n+2}}}{h_a^{F_{n+1}}} = \sum_{cyc}\frac{a^{F_n+F_{n+1}}b^{F_{n+2}}}{(ah_a)^{F_{n+1}}} = \frac{1}{(2F)^{F_{n+1}}}\sum_{cyc}(ab)^{F_{n+2}} \overset{\text{Radon}}{\geq}$$

$$\overset{\text{Radon}}{\geq} \frac{1}{2^{F_{n+1}}F^{F_{n+1}}} \cdot \left(\sum_{cyc} ab\right)^{F_{n+2}} \overset{\text{Gordon}}{\geq} \frac{1}{2^{F_{F_{n+1}}}F^{F_{n+1}}3^{F_{n+2}-1}} \cdot (4F\sqrt{3})^{F_{n+2}} =$$

$$= 2^{F_n+F_{n+2}}(\sqrt{3})^{2-F_{n+2}}F^{F_n}$$

□

Application 2. In any triangle ABC with usual notations and the area F the following inequality holds:

$$\frac{a^{L_n}b^{L_{n+2}}}{h_a^{L_{n+1}}} + \frac{b^{L_n}c^{L_{n+2}}}{h_b^{L_{n+1}}} + \frac{c^{L_n}a^{L_{n+2}}}{h_c^{L_{n+1}}} \geq 2^{L_n+L_{n+2}}(\sqrt{3})^{2-L_{n+2}}F^{L_n}$$

Proof.

$$\sum_{cyc}\frac{a^{L_n}b^{L_{n+2}}}{h_a^{L_{n+1}}} = \sum_{cyc}\frac{a^{L_n+L_{n+1}}b^{L_{n+2}}}{(ah_a)^{L_{n+1}}} = \frac{1}{(2F)^{L_{n+1}}}\sum_{cyc}(ab)^{L_{n+2}} \overset{\text{Radon}}{\geq}$$

$$\overset{\text{Radon}}{\geq} \frac{1}{2^{L_{n+1}}F^{L_{n+1}}} \cdot \frac{1}{3^{L_{n+2}-1}} \cdot \left(\sum_{cyc} ab\right)^{L_{n+2}} \overset{\text{Gordon}}{\geq} \frac{1}{2^{L_{n+1}}F^{L_{n+1}}3^{L_{n+2}-1}} \cdot (4F\sqrt{3})^{L_{n+2}} =$$

$$= 2^{L_n+L_{n+2}}(\sqrt{3})^{2-L_{n+2}}F^{L_n}$$

□

Application 3. In any triangle ABC with usual notations and the area F the following inequality holds:

$$\frac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}}+\frac{b^{F_n^2}c^{F_{2n+1}}}{h_b^{F_{n+1}^2}}+\frac{c^{F_n^2}a^{F_{2n+1}}}{h_c^{F_{n+1}^2}}\geq 2^{F_{2n+1}+F_n^2}(\sqrt{3})^{2-F_{2n+1}}F^{F_n^2}$$

Proof.

$$\sum_{cyc}\frac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}}=\sum_{cyc}\frac{a^{F_n^2+F_{n+1}^2}b^{F_{2n+1}}}{(ah_a)^{F_{n+1}^2}}=\frac{1}{(2F)^{F_{n+1}^2}}\sum_{cyc}(ab)^{F_{2n+1}}\overset{\text{Radon}}{\geq}$$

$$\overset{\text{Radon}}{\geq}\frac{1}{2^{F_{n+1}^2}F^{F_{n+1}^2}}\cdot\frac{1}{3^{F_{2n+1}-1}}\cdot\left(\sum_{cyc}ab\right)^{F_{2n+1}}\overset{\text{Gordon}}{\geq}\frac{1}{2^{F_{n+1}^2}F^{F_{n+1}^2}3^{F_{2n+1}-1}}\cdot(4F\sqrt{3})^{F_{2n+1}}=$$

$$=2^{F_{2n+1}+F_n^2}(\sqrt{3})^{2-F_{2n+1}}F^{F_n^2}$$

□

Application 4.

a. If $a,b,c\in\mathbb{R}_+^*$ such that $abc=1$, then the following inequality is true

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}};$$

b. If $a,b,c\in\mathbb{R}_+^*$ such that $ab+bc+ca=3$, then is true

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}};$$

c. If $a,b,c\in\mathbb{R}_+^*$ such that $a+b+c=3$, then is true

$$\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}+\frac{1}{b^3(F_n^2c+F_{n+1}^2a)}+\frac{1}{c^3(F_n^2a+F_{n+1}^2b)}\geq\frac{3}{F_{2n+1}}$$

Proof.

a. $$\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{\text{Bergströng}}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$$

$$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)}=\frac{ab+bc+ca}{F_{n+2}}\geq\frac{3\cdot\sqrt[3]{(abc)^2}}{F_{n+2}}=\frac{3}{F_{n+2}}.$$

b. $$\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{\text{Bergström}}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$$

(1) $$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1}(ab+bc+ca)}=\frac{3}{(abc)^2F_{n+2}}$$

(2)

But, $ab+bc+ca\geq 3\cdot\sqrt[3]{(abc)^2}\Leftrightarrow(ab+bc+ca)^3\geq 27(abc)^2\Leftrightarrow(abc)^2\leq 1\Leftrightarrow\dfrac{1}{abc}\geq 1$

From (1) and (2) yields the desired inequality.

c. $$\sum_{cyc}\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n^2+acF_{n+1}^2}\overset{\text{Bergström}}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n^2+acF_{n+1}^2)}=$$

$$= \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(F_n^2+F_{n+1}^2)(ab+bc+ca)} = \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{F_{2n+1}(ab+bc+ca)} \overset{\text{Bergström}}{\geq} \frac{\left(\frac{9}{a+b+c}\right)^2}{F_{2n+1}(ab+bc+ca)} \geq$$

$$\geq \frac{9}{F_{2n+1}\cdot\frac{(a+b+c)^2}{3}} = \frac{3}{F_{2n+1}}$$

Remarks.
If $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$, then

$$\sum_{cyc} \frac{1}{a^3(b+c)} \geq \frac{3}{2}, \text{ (IMO - 1995 - Toronto - Canada)}$$

The problem from above show that the inequality also occurs if $a, b, c \in \mathbb{R}_+^*$ such that $a+b+c=3$ and if $a, b, c \in \mathbb{R}_+^*$ such that $ab+bc+ca=3$. □

Application 5.
If $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$, then holds:

$$\frac{1}{a^3(F_nb+F_{n+1}c)} + \frac{1}{b^3(F_nc+F_{n+1}a)} + \frac{1}{c^3(F_na+F_{n+1}b)} \geq \frac{3}{F_{n+2}}$$

Proof.

$$\sum_{cyc} \frac{1}{a^3(F_nb+F_{n+1}c)} = \sum_{cyc} \frac{\frac{1}{a^2}}{abF_n+acF_{n+1}} \overset{\text{Bergström}}{\geq} \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})} =$$

$$= \frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)} = \frac{ab+bc+ca}{F_{n+2}} \geq \frac{3\cdot\sqrt[3]{(abc)^2}}{F_{n+2}} = \frac{3}{F_{n+2}}.$$

□

Application 6.
If $a, b, c \in \mathbb{R}_+^*$ such that $ab+bc+ca = 3$, then is true

$$\frac{1}{a^3(F_nb+F_{n+1}c)} + \frac{1}{b^3(F_nc+F_{n+1}a)} + \frac{1}{c^3(F_na+F_{n+1}b)} \geq \frac{3}{F_{n+2}}$$

Proof.

$$\sum_{cyc} \frac{1}{a^3(F_nb+F_{n+1}c)} = \sum_{cyc} \frac{\frac{1}{a^2}}{abF_n+acF_{n+1}} \overset{\text{Bergström}}{\geq} \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})} =$$

$$= \frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)} = \frac{3}{(abc)^2F_{n+2}} \tag{1}$$

(2)
But, $ab+bc+ca \geq 3\cdot\sqrt[3]{(abc)^2} \Leftrightarrow (ab+bc+ca)^3 \geq 27(abc)^2 \Leftrightarrow (abc)^2 \leq 1 \Leftrightarrow \frac{1}{abc} \geq 1$

From (1) and (2) yields the desired inequality. □

Application 7.
If $a, b, c \in \mathbb{R}_+^*$ such that $a+b+c = 3$, then

$$\frac{1}{a^3(F_n^2b+F_{n+1}^2c)} + \frac{1}{b^3(F_n^2c+F_{n+1}^2a)} + \frac{1}{c^3(F_n^2a+F_{n+1}^2b)} \geq \frac{3}{F_{2n+1}}$$

Proof.

$$\sum_{cyc}\frac{1}{a^3(F_n^2 b+F_{n+1}^2 c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n^2+acF_{n+1}^2}\overset{\text{Bergström}}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n^2+acF_{n+1}^2)}=$$

$$=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(F_n^2+F_{n+1}^2(ab+bc+ca)}=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{F_{2n+1}(ab+bc+ca)}\overset{\text{Bergström}}{\geq}\frac{\left(\frac{9}{a+b+c}\right)^2}{F_{2n+1}(ab+bc+ca)}\geq$$

$$\geq\frac{9}{F_{2n+1}\cdot\frac{(a+b+c)^2}{3}}=\frac{3}{F_{2n+1}}$$

□

Remark.
If $a,b,c\in\mathbb{R}_+^*$ such that $abc=1$, then

$$\sum_{cyc}\frac{1}{a^3(b+c)}\geq\frac{3}{2},\ \text{(IMO - 1995 - Toronto - Canada)}$$

The inequality also occurs if $a,b,c\in\mathbb{R}_+^*$ such that $a+b+c=3$ and if $a,b,c\in\mathbb{R}_+^*$ such that $ab+bc+ca=3$.

Application 8.
If $m>1,n\in\mathbb{N}^*$ then

$$\sum_{k=1}^{n}(1+L_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}(L_nL_{n+1}-2)$$

Proof.

We consider the function $f:\mathbb{R}_+^*\to\mathbb{R}_+^*, f(x)=m^m(1+x)^{m+1}-(m+1)^{m+1}x$ with $f'(x)=(m+1)((m+mx)^m-(m+1)^{m+1})$ and $f''(x)=(m+1)\cdot m^{m+1}(1+x)^{m-1}>0$, $\forall x\in\mathbb{R}_+^*$. Therefore, f is convex and it has the minimum point $x_0=\frac{1}{m}$.

$$\text{So, } f(x)\geq f\left(\frac{1}{m}\right)=0,\ \text{i.e.}$$

$$(1)\qquad m^m(1+x)^{m+1}\geq(m+1)^{m+1}x\Leftrightarrow(1+x)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^2m}x^2$$

From (1) we get $(1+L_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^2m}L_k^2,\forall k\in\mathbb{N}^*$, so we obtain that

$$\sum_{k=1}^{n}(1+L_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}\sum_{k=1}^{n}L_k^2$$

and taking acoount by $\sum_{k=1}^{n}L_k^2=L_nL_{n+1}-2$ we obtain the desired inequality.

□

Application 9.

If $m>1,n\in\mathbb{N}^*$ then $\sum_{k=1}^{n}(1+F_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}F_nF_{n+1}$

Proof.
We consider the function $f:\mathbb{R}_+^*\to\mathbb{R}_+^*, f(x)=m^m(1+x)^{m+1}-(m+1)^{m+1}x$ with $f'(x)=(m+1)((m+mx)^m-(m+1)^{m+1})$ and
$f''(x)=(m+1)\cdot m^{m+1}(1+x)^{m-1}>0, \forall x\in\mathbb{R}_+^*$.
Therefore, f is convex and it has the minimum point $x_0=\frac{1}{m}$. So, $f(x)\geq f(\frac{1}{m})=0$, i.e.

$$(1)\qquad m^m(1+x)^{m+1}\geq(m+1)^{m+1}x\Leftrightarrow(1+x)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}x^2$$

From (1) we get $(1+F_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}F_k^2, \forall k\in\mathbb{N}^*$, so we obtain that

$$\sum_{k=1}^{n}(1+F_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}\sum_{k=1}^{n}F_k^2$$

and taking account by $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$ we obtain the desired inequality

□

Application 10.
If $x,y,z>0$, then

$$\frac{x^2}{(5F_{2n}^2y+2z)(5F_{2n}^2z+2y)}+\frac{y^2}{(5F_{2n}^2z+2x)(5F_{2n}^2x+2z)}+$$
$$+\frac{z^2}{(5F_{2n}^2x+2y)(5F_{2n}^2y+2x)}\geq\frac{3}{L_{4n}^2}$$

Proof.

$$(1)\;\; 5F_{2n}^2+2=5\cdot\frac{1}{5}(\alpha^{2n}-\beta^{2n})^2+2=\alpha^{4n}+\beta^{4n}-2(\alpha\beta)^{2n}+2=\alpha^{4n}+\beta^{4n}=L_{4n}$$

$$\frac{x^2}{(5F_{2n}^2y+2z)(5F_{2n}^2z+2y)}=\frac{x^2}{25F_{2n}^4yz+10F_{2n}^2y^2+10F_{2n}^2z^2+4yz}=$$
$$=\frac{x^2}{(25F_{2n}^4+4)yz+10F_{2n}^2(y^2+z^2)}\overset{\text{PM-GM}}{\geq}\frac{x^2}{(25F_{2n}^4+4)\cdot\frac{y^2+z^2}{2}+10F_{2n}^2(y^2+z^2)}$$

$$(2)\qquad=\frac{2x^2}{y^2+z^2}\cdot\frac{1}{25F_{2n}^4+10F_{2n}^2+4}=\frac{2x^2}{y^2+z^2}\cdot\frac{1}{(5F_{2n}^2+2)^2}\overset{(1)}{=}\frac{2x^2}{y^2+z^2}\cdot\frac{1}{L_{4n}^2}$$

And other two similar inequality., i.e.

$$(3)\qquad\frac{y^2}{(5F_{2n}^2z+2x)(5F_{2n}^2x+2z)}\geq\frac{2y^2}{z^2+x^2}\cdot\frac{1}{L_{4n}^2}$$

$$(4)\qquad\frac{z^2}{(5F_{2n}^2x+2y)(5F_{2n}^2y+2x)}\geq\frac{2z^2}{x^2+y^2}\cdot\frac{1}{L_{4n}^2}$$

Adding up the inequalities $(2),(3)$ and (4) and taking account by Nesbitt - Ionescu inequality (i.e. $\sum_{cyc}\frac{a}{b+c}\geq\frac{3}{2}$ for any $a,b,c>0$) we obtain:

$$\frac{x^2}{(5F_{2n}^2y+2z)(5F_{2n}^2z+2y)}+\frac{y^2}{(5F_{2n}^2z+2x)(5F_{2n}^2x+2z)}+\frac{z^2}{(5F_{2n}^2x+2y)(5F_{2n}^2y+2x)}\geq$$

$$\geq \frac{2x^2}{y^2+z^2} \cdot \frac{1}{L_{4n}^2} \cdot \frac{2y^2}{z^2+x^2} \cdot \frac{1}{L_{4n}^2} + \frac{2z^2}{x^2+y^2} \cdot \frac{1}{L_{4n}^2} \overset{\text{Nesbitt - Ionescu}}{\geq} 2 \cdot \frac{3}{2} \cdot \frac{1}{L_{4n}^2} = \frac{3}{L_{4n}^2}$$

□

Application 11.
If ABC is a triangle a, b, c the lengths of the sides, R the lengths of circumradius, r the lengths of the inradius and s the semiperimeter, then:

$$\left(\frac{F_n^2 a^2 + F_{n+1}^2 b^2}{c}\right)^2 + \left(\frac{F_n^2 b^2 + F_{n+1}^2 c^2}{a}\right)^2 + \left(\frac{F_n^2 c^2 + F_{n+1}^2 a^2}{b}\right)^2 \geq 2F_{2n+1}^2(s^2 - r^2 - 4Rr),$$

$$\forall n \in \mathbb{N}^*$$

Proof.
By Bergström's inequality and the formula $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ we have

$$\left(\frac{F_n^2 a^2 + F_{n+1}^2 b^2}{c}\right)^2 + \left(\frac{F_n^2 b^2 + F_{n+1}^2 c^2}{a}\right)^2 + \left(\frac{F_n^2 c^2 + F_{n+1}^2 a^2}{b}\right)^2 = \sum_{cyc}\left(\frac{F_n^2 a^2 + F_{n+1}^2 b^2}{c}\right)^2 =$$

$$= \sum_{cyc} \frac{(F_n^2 a^2 + F_{n+1}^2 b^2)^2}{c^2} \overset{\text{Bergström}}{\geq} \frac{\left(\sum_{cyc}(F_n^2 a^2 + F_{n+1}^2 b^2)\right)^2}{a^2 + b^2 + c^2} =$$

$$= \frac{(F_n^2 + F_{n+1}^2)^2 (a^2 + b^2 + c^2)^2}{a^2 + b^2 + c^2} = 2F_{2n+1}^2(s^2 - r^2 - 4Rr)$$

□

Application 12.
If ABC be a triangle with a, b, c the lengths of the sides, R the lengths of circumradius, r the lengths of the inradius and s the semiperimeter, then

$$\left(\frac{F_n a^2 + F_{n+1} b^2}{c}\right)^2 + \left(\frac{F_n b^2 + F_{n+1} c^2}{a}\right)^2 + \left(\frac{F_n c^2 + F_{n+1} a^2}{b}\right)^2 \geq 2F_{n+2}^2(s^2 - r^2 - 4Rr),$$

$$\forall n \in \mathbb{N}^*.$$

Proof.
By Bergström inequality and the formula $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ we have

$$\left(\frac{F_n a^2 + F_{n+1} b^2}{c}\right)^2 + \left(\frac{F_n b^2 + F_{n+1} c^2}{a}\right)^2 + \left(\frac{F_n c^2 + F_{n+1} a^2}{b}\right)^2 = \sum_{cyc}\left(\frac{F_n a^2 + F_{n+1} b^2}{c}\right)^2 =$$

$$= \sum_{cyc} \frac{(F_n a^2 + F_{n+1} b^2)^2}{c^2} \overset{\text{Bergström}}{\geq} \frac{\left(\sum_{cyc}(F_n a^2 + F_{n+1} b^2)\right)^2}{a^2 + b^2 + c^2} =$$

$$= \frac{(F_n + F_{n+1})^2 (a^2 + b^2 + c^2)^2}{a^2 + b^2 + c^2} = 2F_{n+2}^2(s^2 - r^2 - 4Rr)$$

□

Application 13.
If $a, b, c \in (0, \frac{\pi}{2})$, then

$$\frac{\tan a}{F_n \sin 2b + F_{n+1} \sin 2c} + \frac{\tan b}{F_n \sin 2c + F_{n+1} \sin 2a} + \frac{\tan c}{F_n \sin 2a + F_{n+1} \sin 2b} > \frac{3}{2F_{n+2}},$$

$$\forall n \in \mathbb{N}^*$$

Proof.
From $a, b, c \in (0, \frac{\pi}{2})$ yields that $\tan a > a, \tan b > b, \tan c > c$ and

$$\sin 2a = 2\sin a \cos a < 2\sin a < 2a, \text{ similarly } \sin 2b < 2b, \sin 2c < 2c.$$

Hence:

$$\frac{\tan a}{F_n \sin 2b + F_{n+1} \sin 2c} + \frac{\tan b}{F_n \sin 2c + F_{n+1} \sin 2a} + \frac{\tan c}{F_n \sin 2a + F_{n+1} \sin 2b} =$$
$$= \sum_{cyc} \frac{\tan a}{F_n \sin 2b + F_{n+1} \sin 2c} > \sum_{cyc} \frac{a}{2bF_n + 2cF_{n+1}} =$$
$$= \frac{1}{2} \sum_{cyc} \frac{a^2}{abF_n + acF_{n+1}} \overset{\text{Bergström}}{\geq} \frac{1}{2} \cdot \frac{(a+b+c)^2}{\sum_{cyc}(abF_n + acF_{n+1})} =$$
$$= \frac{1}{2} \cdot \frac{(a+b+c)^2}{(F_n + F_{n+1})(ab+bc+ca)} = \frac{(a+b+c)^2}{2F_{n+2}(ab+bc+ca)} \geq$$
$$\geq \frac{3(ab+bc+ca)}{2F_{n+2}(ab+bc+ca)} = \frac{3}{2F_{n+2}}$$

□

Application 14.
If $a, b, c \in (0, \frac{\pi}{2})$, then

$$\frac{\tan a}{F_n^2 \sin 2b + F_{n+1}^2 \sin 2c} + \frac{\tan b}{F_n^2 \sin 2c + F_{n+1}^2 \sin 2a} + \frac{\tan c}{F_n^2 \sin 2a + F_{n+1}^2 \sin 2b} > \frac{3}{2F_{2n+1}},$$
$$, \forall n \in \mathbb{N}^*$$

Proof.
From $a, b, c \in (0, \frac{\pi}{2})$ yields that $\tan a > a, \tan b > b, \tan c > c$ and

$$\sin 2a = 2\sin a \cos a < 2\sin a < 2a, \text{ similarly } \sin 2b < 2b, \sin 2c < 2c$$

Hence:

$$\frac{\tan a}{F_n^2 \sin 2b + F_{n+1}^2 \sin 2c} + \frac{\tan b}{F_n^2 \sin 2c + F_{n+1}^2 \sin 2a} + \frac{\tan c}{F_n^2 \sin 2a + F_{n+1}^2 \sin 2b} =$$
$$= \sum_{cyc} \frac{\tan a}{F_n^2 \sin 2b + F_{n+1}^2 \sin 2c} > \sum_{cyc} \frac{a}{2bF_n^2 + 2cF_{n+1}^2} =$$
$$= \frac{1}{2} \sum_{cyc} \frac{a^2}{abF_n^2 + acF_{n+1}^2} \overset{\text{Bergström}}{\geq} \frac{1}{2} \cdot \frac{(a+b+c)^2}{\sum_{cyc}(abF_n^2 + acF_{n+1}^2)} =$$
$$= \frac{1}{2} \cdot \frac{(a+b+c)^2}{(F_n^2 + F_{n+1}^2)(ab+bc+ca)} = \frac{(a+b+c)^2}{2F_{2n+1}(ab+bc+ca)} \geq$$
$$\geq \frac{3(ab+bc+ca)}{2F_{2n+1}(ab+bc+ca)} = \frac{3}{2F_{2n+1}}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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