ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sin(6^\circ) = 4\sin(12^\circ).\sin(18^\circ).\sin(24^\circ)$$

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$$\sin(\pi - x) = \sin x \rightarrow \sin\left(\pi - \frac{2\pi}{5}\right) = \sin\frac{2\pi}{5}$$

$$\sin\frac{3\pi}{5} = \sin\frac{2\pi}{5} \rightarrow \sin\frac{\pi}{5}\left(3 - 4\sin^2\frac{\pi}{5}\right) = 2\sin\frac{\pi}{5}\cos\frac{\pi}{5}$$

$$3 - 4\sin^2\frac{\pi}{5} = 2\cos\frac{\pi}{5} \rightarrow 3 - 4\left(1 - \cos^2\frac{\pi}{5}\right) = 2\cos\frac{\pi}{5}$$

$$4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0 \rightarrow \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$$

$$1 - 2\sin^2\frac{\pi}{10} = \frac{\sqrt{5}+1}{4} \rightarrow \sin^2\frac{\pi}{10} = \frac{3-\sqrt{5}}{8} \rightarrow \sin^2\frac{\pi}{10} = \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} \rightarrow \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

We must prove that:

$$4\sin 12^\circ \sin 18^\circ \sin 24^\circ = \sin 6^\circ \Leftrightarrow$$

$$\Leftrightarrow 8\cos 6^\circ\, 4\sin 12^\circ \sin 18^\circ \sin 24^\circ = 2\cos 6^\circ \sin 6^\circ \Leftrightarrow$$

$$\Leftrightarrow 8\cos 6^\circ \sin 12^\circ \sin 18^\circ \sin 24^\circ = \sin 12^\circ \Leftrightarrow 8\cos 6^\circ \sin 18^\circ \sin 24^\circ = 1 \Leftrightarrow$$

$$\Leftrightarrow \sin 24^\circ \cos 6^\circ \sin 18^\circ = \frac{1}{8} \Leftrightarrow \frac{1}{2}\big(\sin(24^\circ + 6^\circ) + \sin(24^\circ - 6^\circ)\big)\sin 18^\circ = \frac{1}{8} \Leftrightarrow$$

$$\Leftrightarrow (\sin 30^\circ + \sin 18^\circ)\sin 18^\circ = \frac{1}{4} \Leftrightarrow \left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\frac{\sqrt{5}-1}{4} = \frac{1}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sqrt{5}+1}{4} \cdot \frac{\sqrt{5}-1}{4} = \frac{1}{4} \Leftrightarrow \frac{5-1}{16} = \frac{1}{4} \Leftrightarrow \frac{1}{4} = \frac{1}{4}$$