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ABOUT THE SPIEKER'S CEVIANS IN TRIANGLE

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ABSTRACT: We consider ABC a triangle with usual notations. We will study the Spieker's cevians in a triangle ABC. Spieker center is obtain as follow: Le be M_1-middle of the side BC,M_2-middle of the side AC,M_3-middle of the side AB. Spieker center of $\triangle$ABC is the center of the circle inscribed in the medial triangle of $\triangle$ABC. ($\Delta M_1M_2M_3$).Notation for Spieker center: S_p ;

RESULTS:

Notation for Spieker center: S_p ; We consider AA_1= p_a Spieker cevian from A (and analogs);

We consider: $\alpha = \measuredangle BAA_1$ and $\beta = \measuredangle CAA_1$; $\alpha+\beta=A$, M_1M_2 , M_2M_3 , M_1M_3-middle lines in triangle ABC , $M_1M_2=\frac{1}{2}c$, $M_2M_3=\frac{1}{2}a$, $M_1M_3=\frac{1}{2}b$

From direct manipulations using Heron formula we obtain: area $\Delta M_1M_2M_3=\frac{1}{4}S$

S = area of triangle ABC.

$$\frac{1}{2}(M_1M_2+M_2M_3+M_1M_3)=\frac{1}{2}\frac{1}{2}(a+b+c)=\frac{1}{4}2p=\frac{1}{2}p.$$

We consider r_1-inradius of the circle inscribed in medial triangle of $\triangle$ABC.

From $\frac{1}{4}pr=\frac{1}{2}pr_1\rightarrow r_1=\frac{1}{2}r$. We consider $\triangle AM_3S_p$, $M_3S_p=\frac{r_1}{\sin\frac{C}{2}}=\frac{r}{2\sin\frac{C}{2}}$ (and analogs)

In $\triangle AM_3S_p$ we use sinus law and obtain: $\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin AM_3S_p}$

$$\measuredangle AM_3M_2=\measuredangle ABC \text{ because } M_2M_3 \parallel BC$$

$\measuredangle M_1M_3M_2=\measuredangle ACB$ because $M_1M_3M_2C$ parallelogram and two angles are opposite angles.

$$\measuredangle AM_3S_p=\measuredangle AM_3M_2+\frac{1}{2}(\measuredangle M_1M_3M_2)=\measuredangle B+\measuredangle\frac{1}{2}C$$

$$\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin AM_3S_p}\rightarrow\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin\left(B+\frac{1}{2}C\right)}$$

$$\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(B+\frac{1}{2}C\right)}\rightarrow\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(\frac{2B+C}{2}\right)}$$

$$A+B+C=\pi\rightarrow B+C=\pi-A\rightarrow 2B+C=\pi+B-A$$

$$\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(\frac{\pi+B-A}{2}\right)}=\frac{AS_p}{\sin\left(\frac{\pi}{2}-\frac{A-B}{2}\right)}$$

We use the well-known formula:$\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and obtain:

$$\sin\left(\frac{\pi}{2}-\frac{A-B}{2}\right)=\cos\frac{A-B}{2},\ \cos\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}\text{ (and analogs) (Karl Mollweide formulas)}$$

We obtain:

$$\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\frac{a+b}{c}\sin\frac{C}{2}}\rightarrow\sin\alpha=\frac{a+b}{c}\,\frac{r}{2AS_p}\ (1)$$

Using same method for $\triangle AM_2S_p$ obtain:

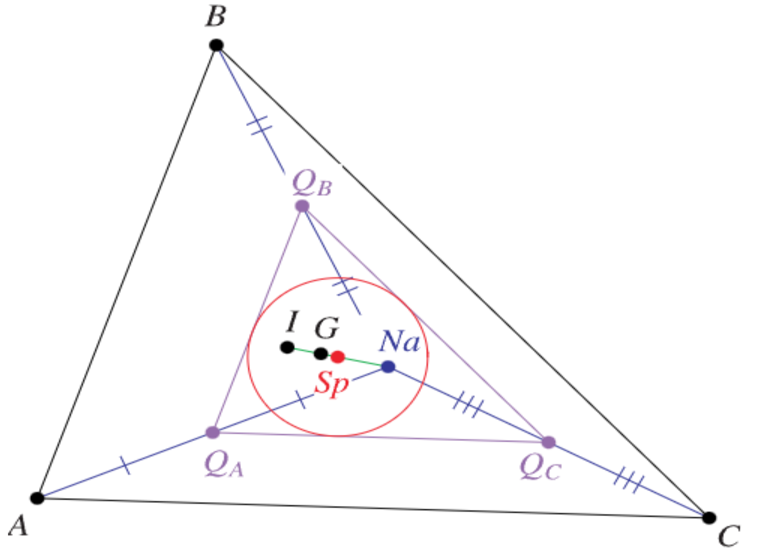
$$\sin\beta=\frac{a+c}{b}\,\frac{r}{2AS_p}\ (2)$$

$$S_{\triangle BAA_1}+S_{\triangle CAA_1}=S=p\,r,\ S_{\triangle BAA_1}=\frac{1}{2}cp_a\sin\alpha=\frac{1}{2}\frac{r(a+b)}{2AS_p}p_a\text{ and}$$

$$S_{\triangle CAA_1}=\frac{1}{2}bp_a\sin\beta=\frac{1}{2}\frac{r(a+c)}{2AS_p}p_a,\ \frac{1}{2}\frac{r}{2AS_p}p_a(a+b+a+c)=p\,r\rightarrow$$

$$p_a=\frac{4p}{2p+a}AS_p\text{ (and analogs) }(3)$$

We will use this theorem: Points I, G,S_P,N_a are colinear, line that passes through these points is called Nagel line. I (incenter), G (triangle centroid), S_P (Spieker center), N_a (Nagel point).[1]



From this we obtain:

$$l_a\le m_a\le p_a\le n_a\text{ (and analogs) }(4)$$

From (3) and (4) →

$$\mathbf{l_a \le m_a \le \frac{4p}{2p+a}\, AS_p \le n_a}\ \textbf{(and analogs) (5)}$$

We obtain

$$\mathbf{AI \le AG \le AS_p \le AN_a}\ \textbf{(6)}$$

AI= $\frac{r}{\sin\frac{A}{2}}$ =4Rsin $\frac{B}{2}\sin\frac{C}{2}$ =$\sqrt{bc-4Rr}$(and analogs), AG= $\frac{2}{3}m_a$(and analogs)

AN_a=$\sqrt{4r^2+(b-c)^2}=\frac{an_a}{p}$(and analogs). From (3), (6) and

$$\mathbf{AN_a=\frac{an_a}{p} \to p_a \le \frac{4a}{2p+a} n_a}\ \textbf{(7)}$$

From (3), (6) and AG= $\frac{2}{3}m_a \to$

$$\mathbf{\frac{8p}{3(2p+a)} m_a \le p_a}\ \textbf{(8)}$$

From (7) after some banal manipulations and summation we obtain

$$\mathbf{\frac{p}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{2} \le \frac{n_a}{p_a}+\frac{n_b}{p_b}}\ \textbf{(9)}$$

$$\mathbf{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}\right) \le 4\left(\frac{n_a}{p_a}+\frac{n_b}{p_b}\right)-2}\ \textbf{(10)}$$

From (8) after some banal manipulation and summation we obtain

$$\mathbf{\frac{8p}{3}\left[\frac{4p+a+b}{(2p+a)(2p+b)}\right] \le \frac{p_a}{m_a}+\frac{p_b}{m_b}}\ \textbf{(11)}$$

Now we will use some proprieties of Nagel line and well-known results:

1)$N_aS_P = S_PI \to AS_P$-median in $\triangle AIN_a$

2)$2N_aI=3N_aG=4S_PI=6GI=12GS_P$

3)$9GI^2=p^2+5r^2-16Rr \to N_aI^2=p^2+5r^2-16Rr$ [1]

$\mathbf{4AS_P{}^2=2(AI^2+A\,N_a{}^2)-N_aI^2=2(AI^2+A\,N_a{}^2)-p^2-5r^2+16Rr}$**(and analogs) (12)**

and an equivalent form

$$\mathbf{4AS_P{}^2=2(b^2+c^2-bc)+8Rr+3r^2-p^2}\ \textbf{(and analogs) (13)}$$

$$\mathbf{4AS_P{}^2=b^2+c^2+(b-c)^2+8Rr+3r^2-p^2}\ \textbf{(and analogs) (14)}$$

From (3), (13), (14),3) we obtain:

$$\mathbf{p_a=\frac{2p}{2p+a}\sqrt{2(b^2+c^2-bc)+8Rr+3r^2-p^2}}\ \textbf{(and analogs) (15)}$$

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{2(AI^2 + A\,N_a{}^2) - p^2 - 5r^2 + 16Rr}}\textbf{(and analogs) (16)}$$

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{b^2 + c^2 + (b-c)^2 + 8Rr + 3r^2 - p^2}}\textbf{ (and analogs) (17)}$$

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{2(AI^2 + A\,N_a{}^2) - 9GI^2}}\textbf{(and analogs) (18)}$$

$4AS_P{}^2=2(a^2+b^2+c^2-bc-a^2)+8Rr+3r^2-p^2=2(a^2+b^2+c^2)-2(a^2+bc)+8Rr+3r^2-p^2$. We use $a^2+b^2+c^2=2p^2-8Rr-2r^2$

$$\mathbf{4AS_P{}^2=3p^2-8Rr-r^2-2(a^2+bc)}\textbf{ (and analogs) (19)}$$

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{3p^2-8Rr-r^2-2(a^2+bc)}}\textbf{ (and analogs) (20)}$$

From (13) and $4m_a^2=2(b^2+c^2)-a^2$ (and analogs) we obtain:

$$\mathbf{4AS_P{}^2=4m_a^2+a^2-2bc+8Rr+3r^2-p^2}\textbf{(and analogs) (21)}$$

From (3) and (21) we obtain:

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{4m_a^2+a^2-2bc+8Rr+3r^2-p^2}}\textbf{(and analogs) (22)}$$

From (21) and $4m_a^2=n_a^2+g_a^2+2r_br_c$ (and analogs) [2] we obtain:

$$\mathbf{4AS_P{}^2=n_a^2+g_a^2+2r_br_c+a^2-2bc+8Rr+3r^2-p^2}\textbf{(and analogs)}$$

We know that: $bc=rr_a+r_br_b$(and analogs) and using (22) we obtain

$$\mathbf{4AS_P{}^2=n_a^2+g_a^2+a^2-2\,rr_a+8Rr+3r^2-p^2}\textbf{(and analogs) (23)}$$

From (23) and (3) we obtain:

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{n_a^2+g_a^2+a^2-2\,rr_a+8Rr+3r^2-p^2}}\textbf{(and analogs) (24)}$$

From (23) and $p^2=n_a^2+2r_ah_a$ (and analogs) we obtain:

$4AS_P{}^2=n_a^2+g_a^2+a^2-2\,rr_a+8Rr+3r^2-n_a^2-2r_ah_a$

$$\mathbf{4AS_P{}^2=g_a^2+a^2+8Rr+3r^2-2\,r_a(h_a+r)}\textbf{(and analogs) (25)}$$

From (3) and (25) we obtain:

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{g_a^2+a^2+8Rr+3r^2-2\,r_a(h_a+r)}}\textbf{(and analogs) (26)}$$

From (19) and $p^2=n_a^2+2r_ah_a$ (and analogs) [3] we obtain:

$4AS_P{}^2=n_a^2+n_b^2+n_c^2+2r_ah_a+2r_bh_b+2r_ch_c-8Rr-r^2-2(a^2+bc)$

(and analogs) (27)

From (3) and (27) we obtain:

$$p_a = \frac{2p}{2p+a}\sqrt{n_a^2 + n_b^2 + n_c^2 + 2r_a h_a + 2r_b h_b + 2r_c h_c - 8Rr - r^2 - 2(a^2 + bc)}$$

(and analogs) (28)

From (19) and $n_a n_b + n_b n_c + n_a n_c \ge p^2$ [4] we obtain:

$$4AS_P^{\ 2} \le 3(n_a n_b + n_b n_c + n_a n_c) - 8Rr - r^2 - 2(a^2 + bc)$$ **(and analogs) (29)**

From (3) and (29) we obtain:

$$p_a \le \frac{2p}{2p+a}\sqrt{3(n_a n_b + n_b n_c + n_a n_c) - 8Rr - r^2 - 2(a^2 + bc)}$$ **(and analogs) (30)**

From (19) and $\sum \sqrt{n_a m_a l_a g_a} \ge p^2$ [5] we obtain:

$$4AS_P^{\ 2} \le 3\sum\sqrt{n_a m_a l_a g_a} - 8Rr - r^2 - 2(a^2 + bc)$$ **(and analogs) (31)**

From (31) and (3) we obtain:

$$p_a \le \frac{2p}{2p+a}\sqrt{3\sum\sqrt{n_a m_a l_a g_a} - 8Rr - r^2 - 2(a^2 + bc)}$$ **(and analogs) (32)**

From $m_a l_a \ge p(p-a)$ (and analogs) (Panaitopol) after summation we obtain:

$m_a l_a + m_b l_b + m_c l_c \ge p^2$ and from (19) we obtain:

$$4AS_P^{\ 2} \le 3(m_a l_a + m_b l_b + m_c l_c) - 8Rr - r^2 - 2(a^2 + bc)$$ **(and analogs) (33)**

From (33) and (3) we obtain:

$$p_a \le \frac{2p}{2p+a}\sqrt{3(m_a l_a + m_b l_b + m_c l_c) - 8Rr - r^2 - 2(a^2 + bc)}$$ **(and analogs) (34)**

From [6]: p_a, p_b, p_c- are sides of a triangle regardless the shape of triangle ABC.

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