ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$, then :

$$\frac{a\mathrm{b}+2a+\mathrm{b}}{a+2\mathrm{b}+1}+\frac{a\mathrm{c}+2a+\mathrm{c}}{a+2\mathrm{c}+1}\ge\frac{4a}{a+1}$$

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Solution 1 by Șerban George Florin-Romania

$$a>0, f:[0,\infty)\to\mathbb{R}, f(x)=\frac{ax+2a+x}{a+2x+1}$$

$$f'(x)=\frac{(a+1)(a+2x+1)-2(ax+2a+x)}{(a+2x+1)^2}$$

$$f'(x)=\frac{a^2+2ax+a+a+2x+1-2ax-4a-2x}{(a+2x+1)^2}$$

$$f'(x)=\frac{a^2-2a+1}{(a+2x+1)^2}=\frac{(a-1)^2}{(a+2x+1)^2}=\left(\frac{a-1}{a+2x+1}\right)^2>0$$

$$\Rightarrow f\uparrow[0,\infty)$$

$$\left.\begin{array}{l} b>0\Rightarrow f(b)>f(0)=\dfrac{2a}{a+1} \\ c>0\Rightarrow f(c)>f(0)=\dfrac{2a}{a+1}\end{array}\right\}\Rightarrow f(b)+f(c)>\frac{2a}{a+1}+\frac{2a}{a+1}=\frac{4a}{a+1}$$

$$\Rightarrow\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\ge\frac{4a}{a+1}$$

$$(\forall)a,b,c>0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a\mathrm{b}+2a+\mathrm{b}}{a+2\mathrm{b}+1}+\frac{a\mathrm{c}+2a+\mathrm{c}}{a+2\mathrm{c}+1}\ge\frac{4a}{a+1}\Leftrightarrow\frac{a\mathrm{b}+2a+\mathrm{b}}{a+2\mathrm{b}+1}-1+\frac{a\mathrm{c}+2a+\mathrm{c}}{a+2\mathrm{c}+1}-1$$

$$\ge\frac{4a}{a+1}-2\Leftrightarrow\frac{a\mathrm{b}+a-\mathrm{b}-1}{a+2\mathrm{b}+1}+\frac{a\mathrm{c}+a-\mathrm{c}-1}{a+2\mathrm{c}+1}\ge\frac{2a-2}{a+1}$$

$$\Leftrightarrow\frac{a(\mathrm{b}+1)-(\mathrm{b}+1)}{a+2\mathrm{b}+1}+\frac{a(\mathrm{c}+1)-(\mathrm{c}+1)}{a+2\mathrm{c}+1}\ge\frac{2(a-1)}{a+1}$$

$$\Leftrightarrow\frac{(a-1)(\mathrm{b}+1)}{a+2\mathrm{b}+1}+\frac{(a-1)(\mathrm{c}+1)}{a+2\mathrm{c}+1}\ge\frac{2(a-1)}{a+1}$$

$$\Leftrightarrow (a-1)\left(\left(\frac{b+1}{a+2b+1}-\frac{1}{a+1}\right)+\left(\frac{c+1}{a+2c+1}-\frac{1}{a+1}\right)\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)}{a+1}\left(\frac{ab+a+b+1-a-2b-1}{a+2b+1}+\frac{ac+a+c+1-a-2c-1}{a+2c+1}\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)}{a+1}\left(\frac{b(a-1)}{a+2b+1}+\frac{c(a-1)}{a+2c+1}\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)^2}{a+1}\left(\frac{b}{a+2b+1}+\frac{c}{a+2c+1}\right)\geq 0 \rightarrow \text{true}$$

$$\therefore \frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq \frac{4a}{a+1}\ \forall a,b,c>0,''='' \text{ iff } a=1\ (\text{QED})$$