ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{k=1}^{n} e^{k\cdot k!} > \frac{n}{\sqrt[n]{e}} \cdot e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}}, \qquad n \geq 2, n \in \mathbb{N}$$

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Lemma 1:

$$1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1, \qquad n \in \mathbb{N}^*$$

Proof: For $n = 1 \to 1\cdot 1! = (1+1)! - 1$. True.

$$P(n): \sum_{k=1}^{n} k\cdot k! = (n+1)! - 1$$

→suppose true

$$P(n+1): \sum_{k=1}^{n+1} k\cdot k! = (n+2)! - 1$$

→to prove

$$\sum_{k=1}^{n+1} k\cdot k! = \sum_{k=1}^{n} k\cdot k! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$$

$$= (n+1)!\,(n+1+1) - 1 = (n+2)! - 1$$

$$P(n) \to P(n+1)$$

Lemma 2:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, \qquad n \in \mathbb{N}$$

Proof:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k$$

For $a = b = 1$:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

Lemma 3:

$$(n+1)! \geq 2^n, \qquad n \in \mathbb{N}$$

Proof: For $n=0\rightarrow 1!>2^0$.True.

$$P(n):(n+1)!\geq 2^n$$

→suppose true

$$P(n+1):(n+2)!\geq 2^{n+1}$$

→to prove

$$(n+2)!=(n+1)!\,(n+2)\geq 2^n(n+2)\geq 2^{n+1}\Leftrightarrow n+2\geq 2$$
$$P(n)\rightarrow P(n+1)$$

Back to the problem:

$$\sum_{k=1}^{n}e^{k\cdot k!}\overset{AM-GM}{\overbrace{>}}n\sqrt[n]{\prod_{k=1}^{n}e^{k\cdot k!}}=n\sqrt[n]{e^{\sum_{k=1}^{n}k\cdot k!}}\overset{Lemma\ 1}{\overbrace{=}}n\sqrt[n]{e^{(n+1)!-1}}=$$

$$=\frac{n}{\sqrt[n]{e}}\cdot\sqrt[n]{e^{(n+1)!}}>\frac{n}{\sqrt[n]{e}}\cdot e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}}\Leftrightarrow$$

$$\Leftrightarrow\sqrt[n]{e^{(n+1)!}}>e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}}\overset{Lemma\ 2}{\overbrace{\Leftrightarrow}}\sqrt[n]{e^{(n+1)!}}>e^{\frac{2^n}{n}}\Leftrightarrow$$

$$\Leftrightarrow\sqrt[n]{e^{(n+1)!}}>\sqrt[n]{e^{2^n}}\Leftrightarrow(n+1)!>2^n$$ **. True by Lemma 3.**