ROMANIAN MATHEMATICAL MAGAZINE

If $x > y > z > 0$, *then prove that* :

$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}\ge 5$$

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$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sqrt{x}+\sqrt{z}\right)^2}{x-y+y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}$$

$$=\frac{x+z+2\sqrt{xz}}{x-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}\therefore \frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}$$

$$\ge\frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8\left(\sqrt{t}-1\right)}\left(t=\frac{x}{z}\right)\to(1)$$

$$\text{Let } f(t)=\frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8\left(\sqrt{t}-1\right)}\ \forall\, t\in(1,\infty)\ \textit{and then}:$$

$$f'(t)=\frac{3t^2-4t.\sqrt{t}-16}{16\sqrt{t}\left(\sqrt{t}-1\right)^2}=\frac{3m^4-4m^3-16}{16m(m-1)^2}\ \left(m=\sqrt{t}\right)$$

$$=\frac{(m-2)(3m^3+2m^2+4m+8)}{16m(m-1)^2}\therefore f'(t)=0 \text{ has } a \text{ unique root, it being } t=4$$

$$(\because 3m^3+2m^2+4m+8>0)\ \textit{and}\ f''(t)|_{t=4}$$

$$=\left.\frac{t.\sqrt{t}(3t^2+8t+48)-t(9t^2+16)}{32t^2.\sqrt{t}\left(\sqrt{t}-1\right)^3}\right|_{t=4}=\frac{3}{8}>0$$

$$\therefore f(t)\ \textit{attains a}\ \text{minima}\ \textit{at}\ t=4\ \textit{and}\ \because f'(t)\le 0\ \forall\, t\in(1,4]\ \textit{and}$$
$$f'(t)\ge 0\ \forall\, t\in[4,\infty)\therefore f(t) \text{ is } \downarrow \text{ on } (1,4]\ \textit{and}\ f(t) \text{ is } \uparrow \text{ on } [4,\infty)\therefore f(t)\ge f(4)=5$$

$$\therefore \frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8\left(\sqrt{t}-1\right)}\ge 5\to(2)\therefore (1)\ \textit{and}\ (2)\Rightarrow$$

$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}\ge 5\ \forall\, x>y>z>0,''='' \text{ iff } x=4z\ \text{(QED)}$$