ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x \geq z$, then prove that :

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}\leq\sqrt{5}$$

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$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}$$

$$=\sqrt{2}.\frac{x}{\sqrt{2(x^2+y^2)}}+\sqrt{2}.\frac{y}{\sqrt{2(y^2+z^2)}}+\sqrt{\frac{z}{z+x}}\overset{\text{CBS}}{\leq}$$

$$\sqrt{2+2+1}.\sqrt{\frac{x^2}{2(x^2+y^2)}+\frac{y^2}{2(y^2+z^2)}+\frac{z}{z+x}}\overset{?}{\leq}\sqrt{5}$$

$$\Leftrightarrow\frac{x^2}{x^2+y^2}+\frac{y^2}{y^2+z^2}+\frac{2z}{z+x}\overset{?}{\leq}2\Leftrightarrow\frac{2x}{z+x}-1\overset{?}{\geq}\frac{x^2y^2+z^2x^2+x^2y^2+y^4}{x^2y^2+z^2x^2+y^4+y^2z^2}-1$$

$$\Leftrightarrow\frac{x-z}{z+x}\overset{?}{\geq}\frac{y^2(x-z)(z+x)}{x^2y^2+z^2x^2+y^4+y^2z^2}$$

$$\Leftrightarrow(x-z)\left(\frac{1}{z+x}-\frac{y^2(z+x)}{x^2y^2+z^2x^2+y^4+y^2z^2}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{(x-z)(x^2y^2+z^2x^2+y^4+y^2z^2-y^2z^2-x^2y^2-2y^2zx)}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{(x-z)(z^2x^2+y^4-2y^2zx)}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{(x-z)(zx-y^2)^2}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0\rightarrow\text{true}\because x,y,z>0\wedge x\geq z$$

$$\therefore\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}\leq\sqrt{5}\;\forall\,x,y,z>0\,\big|\,x\geq z\;(\text{QED})$$