ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x+y\geq 0, x-y\geq 0 \text{ and } \sqrt{\left(\frac{x+y}{2}\right)^3}+\sqrt{\left(\frac{x-y}{2}\right)^3}=27,$$

$$\text{then prove that}: x\geq 9$$

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$$\sqrt{\frac{x+y}{2}}^3+\sqrt{\frac{x-y}{2}}^3=27$$

$$\Rightarrow\left(\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}\right)\left(\frac{x+y}{2}+\frac{x-y}{2}-\sqrt{\frac{x^2-y^2}{4}}\right)=27$$

$$\Rightarrow\sqrt{\frac{x+y}{2}+\frac{x-y}{2}+2\sqrt{\frac{x^2-y^2}{4}}}.\left(x-\sqrt{\frac{x^2-y^2}{4}}\right)=27$$

$$\Rightarrow\sqrt{x+2m}.(x-m)=27\to(1)\ \left(m=\sqrt{\frac{x^2-y^2}{4}}\right)$$

$$\text{Now}, 27=\sqrt{\left(\frac{x+y}{2}\right)^3}+\sqrt{\left(\frac{x-y}{2}\right)^3}\overset{A-G}{\geq}2.\sqrt{\sqrt{\frac{x^2-y^2}{4}}^3}=2\sqrt{m^3}\Rightarrow\frac{729}{4}\geq m^3$$

$$\Rightarrow m\leq\frac{9}{\sqrt[3]{4}}<9\Rightarrow 9-m>0\to(2)$$

$$\text{Now, we assume } x<9 \text{ and then}: \sqrt{x+2m}.(x-m)<\sqrt{x+2m}.(9-m)$$

$$<\sqrt{9+2m}.(9-m)\ (\because 9-m>0\ via\ (2))\overset{via\ (1)}{\Rightarrow}\boxed{27\overset{(*)}{<}\sqrt{9+2m}.(9-m)}$$

$$\text{Let } f(m)=\sqrt{9+2m}.(9-m)\ \forall\ m\in\left[0,\frac{9}{\sqrt[3]{4}}\right]$$

$$\left(\because x+y\geq 0, x-y\geq 0\Rightarrow m=\sqrt{\frac{x^2-y^2}{4}}\geq 0\right) \text{ and then}: f'(m)=\frac{-3m}{\sqrt{9+2m}}$$

$$\leq 0\ \forall\ m\in\left[0,\frac{9}{\sqrt[3]{4}}\right]\Rightarrow f(m) \text{ is } \downarrow \text{ on } \left[0,\frac{9}{\sqrt[3]{4}}\right]\ \therefore f(m)\leq f(0)=27$$

$$\Rightarrow\boxed{\sqrt{9+2m}.(9-m)\leq 27} \text{ which is a contradiction to } (*)$$

$$\Rightarrow \text{our assumption is incorrect} \therefore x\geq 9\ (\text{QED})$$