ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$ and $x + y \leq 1$, then prove that :

$$\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\geq 2\sqrt{5}-\frac{4}{5}$$

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$$\sqrt{4x^2+\frac{1}{x^2}}=\sqrt{4x^2+\frac{1}{4x^2}+\frac{1}{4x^2}+\frac{1}{4x^2}+\frac{1}{4x^2}}\overset{A-G}{\geq}$$

$$\sqrt{5.\sqrt[5]{4x^2.\frac{1}{4x^2}.\frac{1}{4x^2}.\frac{1}{4x^2}.\frac{1}{4x^2}}}\Rightarrow\sqrt{4x^2+\frac{1}{x^2}}\geq\frac{\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{x^{\frac{3}{5}}}\right)\text{ and similarly,}$$

$$\sqrt{4y^2+\frac{1}{y^2}}\geq\frac{\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{y^{\frac{3}{5}}}\right)\therefore\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}\geq\frac{\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{x^{\frac{3}{5}}}+\frac{1}{y^{\frac{3}{5}}}\right)$$

$$\overset{Jensen}{\geq}\frac{2\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{\left(\frac{x+y}{2}\right)^{\frac{3}{5}}}\right)\left(\because f(t)=\frac{1}{t^{\frac{3}{5}}}\ \forall\, t\in(0,1)\text{ is convex as } f''(t)=\frac{24}{25t^{\frac{13}{5}}}>0\right)$$

$$\overset{via\ (1)}{\geq}\frac{2\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{\left(\frac{1}{2}\right)^{\frac{3}{5}}}\right)=2\sqrt{5}\therefore\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}\geq 2\sqrt{5}\rightarrow(1)$$

$$\text{Again, }\frac{x}{x^2+1}=\frac{x}{x^2+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\overset{A-G}{\leq}\frac{x}{5.\sqrt[5]{x^2.\frac{1}{4}.\frac{1}{4}.\frac{1}{4}.\frac{1}{4}}}\Rightarrow\frac{x}{x^2+1}\leq\frac{2^{\frac{8}{5}}}{5}.\left(x^{\frac{3}{5}}\right)$$

$$\text{and similarly, }\frac{y}{y^2+1}\leq\frac{2^{\frac{8}{5}}}{5}.\left(y^{\frac{3}{5}}\right)\therefore\frac{x}{x^2+1}+\frac{y}{y^2+1}\leq\frac{2^{\frac{8}{5}}}{5}.\left(x^{\frac{3}{5}}+y^{\frac{3}{5}}\right)\overset{Jensen}{\leq}$$

$$2.\frac{2^{\frac{8}{5}}}{5}.\left(\frac{x+y}{2}\right)^{\frac{3}{5}}\left(\because F(t)=t^{\frac{3}{5}}\ \forall\, t\in(0,1)\text{ is concave as } F''(t)=-\frac{6}{25t^{\frac{7}{5}}}<0\right)$$

$$=2.\frac{2^{\frac{8}{5}}}{5}.\left(\frac{1}{2}\right)^{\frac{3}{5}}=\frac{4}{5}\Rightarrow-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\geq-\frac{4}{5}\rightarrow(2)$$

$$\therefore(1)+(2)\Rightarrow\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\geq 2\sqrt{5}-\frac{4}{5},$$

$$''=''\text{ iff } x=y=\frac{1}{2}\ \text{(QED)}$$