ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$, then prove that:

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}$$

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$$\frac{x^4+y^4}{(x+y)^4}-\frac{1}{8}=\frac{7x^4+7y^4-4x^3y-4xy^3-6x^2y^2}{8(x+y)^4}$$

$$=\frac{3(x^2-y^2)^2+4\left(x^4+y^4-xy(x^2+y^2)\right)}{8(x+y)^4}$$

$$\geq\frac{3(x^2-y^2)^2+4\left(\frac{(x^2+y^2)^2}{2}-xy(x^2+y^2)\right)}{8(x+y)^4}$$

$$=\frac{3(x^2-y^2)^2+2(x^2+y^2)(x-y)^2}{8(x+y)^4}\geq\frac{3(x-y)^2(x+y)^2+(x+y)^2(x-y)^2}{8(x+y)^4}$$

$$=\frac{(x-y)^2}{2(x+y)^2}\Rightarrow\frac{x^4+y^4}{(x+y)^4}-\frac{1}{8}+\frac{\sqrt{xy}}{x+y}\overset{G-H}{\geq}\frac{(x-y)^2}{2(x+y)^2}+\frac{4xy}{2(x+y)^2}=\frac{(x+y)^2}{2(x+y)^2}=\frac{1}{2}$$

$$\Rightarrow\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{1}{8}+\frac{1}{2}=\frac{5}{8}\ \forall\, x,y>0,\text{''}=\text{''}\ \text{iff } x=y\ (\text{QED})$$