ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x < y < z$, then prove that :

$$\frac{x^3 z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2 z}\geq 12$$

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$$\frac{x^3 z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2 z}\geq 12$$

$$\Leftrightarrow \frac{\left(\frac{x}{z}\right)^3}{\left(\frac{y}{z}\right)^2\left(\frac{x}{z}+\left(\frac{y}{z}\right)^2\right)}+\frac{\left(\frac{y}{z}\right)^4}{\frac{x}{z}+\left(\frac{y}{z}\right)^2}+\frac{1+15\left(\frac{x}{z}\right)^3}{\left(\frac{x}{z}\right)^2}\geq 12$$

$$\Leftrightarrow \frac{a^3}{b^2(a+b^2)}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12\ \left(\frac{x}{z}=a,\frac{y}{z}=b\right)$$

$$\Leftrightarrow \frac{(a^3+b^6)-b^6}{b^2(a+b^2)}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12 \Leftrightarrow$$

$$\frac{a^2+b^4-ab^2}{b^2}-\frac{b^4}{a+b^2}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12 \Leftrightarrow \frac{a^2}{b^2}+b^2+\frac{1}{a^2}+14a\overset{(*)}{\geq} 12$$

$$\text{Now,} \frac{a^2}{b^2}+b^2+\frac{1}{a^2}+14a\overset{A-G}{\geq} 2.\sqrt{\frac{a^2}{b^2}.b^2}+\frac{1}{a^2}+14a=\frac{1}{a^2}+16a\overset{?}{\geq} 12$$

$$\Leftrightarrow 16a^3-12a^2+1\overset{?}{\geq} 0 \Leftrightarrow (4a+1)(2a-1)^2\overset{?}{\geq} 0 \to \text{true} \because a=\frac{x}{z}>0$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{x^3 z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2 z}\geq 12 \text{ for } 0<x<y<z,$$

$$''='' \text{ iff } a=b^2 \wedge a=\frac{1}{2} \therefore ''='' \text{ iff } a=\frac{1}{2}, b=\frac{1}{\sqrt{2}}$$

$$\text{and so,} ''='' \text{ iff } z=k, x=\frac{k}{2}, y=\frac{k}{\sqrt{2}}\ \forall k>0 \text{ (QED)}$$