ROMANIAN MATHEMATICAL MAGAZINE

If $1+a+b\neq 0$, then :

$$a^2+b^2+\left(\frac{6-a-b-ab}{1+a+b}\right)^2\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2+\left(\frac{6-a-b-ab}{1+a+b}\right)^2=|a|^2+|b|^2+\left|\frac{6-a-b-ab}{1+a+b}\right|^2$$

$$\geq\frac{1}{3}\left(|a|+|b|+\left|\frac{6-a-b-ab}{1+a+b}\right|\right)^2\overset{?}{\geq}3\Leftrightarrow\boxed{|a|+|b|+\left|\frac{6-a-b-ab}{1+a+b}\right|\underset{(*)}{\overset{?}{\geq}}3}$$

Now, triangle inequality $\Rightarrow\forall\, a,b\in\mathbb{C}\,\big|\,1+a+b\neq 0$, we have :

$$|a|+|b|+\left|\frac{6-a-b-ab}{1+a+b}\right|\geq\left|a+b+\frac{6-a-b-ab}{1+a+b}\right|\overset{?}{\geq}3$$

$$\Leftrightarrow\left|\frac{a+b+(a+b)^2+6-a-b-ab}{1+a+b}\right|\overset{?}{\geq}3$$

$$\Leftrightarrow\boxed{\left|a^2+b^2+ab+6\right|\underset{(**)}{\overset{?}{\geq}}3|1+a+b|}$$

$$\text{Now, } a^2+b^2+ab+6\geq\frac{3}{4}(a+b)^2+6\geq 6>0\ \forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0$$

$$\therefore\left|a^2+b^2+ab+6\right|=a^2+b^2+ab+6\geq\frac{3}{4}(a+b)^2+6$$

$$\forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0\to(1)$$

Also, $\forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0$, we have via triangle inequality,

$$3|1+a+b|\leq 3(1+|a+b|)\to(2)\therefore(1),(2)$$

$\Rightarrow$ in order to prove $(**)$ $\forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0$, it suffices to prove :

$$\frac{3}{4}(a+b)^2+6\geq 3(1+|a+b|)\Leftrightarrow|a+b|^2+8\overset{?}{\geq}4+4|a+b|$$

$$\Leftrightarrow|a+b|^2-4|a+b|+4\overset{?}{\geq}0\Leftrightarrow(|a+b|-2)^2\overset{?}{\geq}0\to\text{true}\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true }\forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0\therefore a^2+b^2+\left(\frac{6-a-b-ab}{1+a+b}\right)^2\geq 3$$

$$\forall\, a,b\in\mathbb{R}\,\big|\,1+a+b\neq 0,''=''\ \text{iff } a=b=1\ (\text{QED})$$