ROMANIAN MATHEMATICAL MAGAZINE

If $\cos a+\cos b+\cos c=0$, then :

$$\cos^2 a+\cos^2 b+\cos^2 c\le 2$$

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$$\cos^2 a+\cos^2 b+\cos^2 c\le 2 \overset{\cos a+\cos b+\cos c=0}{\Leftrightarrow}$$

$$\cos^2 a+\cos^2 b+(-\cos a-\cos b)^2\le 2\Leftrightarrow 2(\cos^2 a+\cos^2 b)+2\cos a\cos b\le 2$$

$$\Leftrightarrow \cos^2 a+\cos^2 b+\cos a\cos b\overset{(*)}{\le} 1$$

$\boxed{\text{Case 1}}$ $\cos a,\cos b\ge 0$ *and then* : LHS of $(*)=(\cos a+\cos b)^2-\cos a\cos b$

$$\overset{\cos a+\cos b+\cos c=0}{=}(-\cos c)^2-\cos a\cos b\le\cos^2 c\ (\because \cos a\cos b\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\boxed{\text{Case 2}}$ $\cos a,\cos b\le 0$ *and then* : LHS of $(*)=(\cos a+\cos b)^2-\cos a\cos b$

$$\overset{\cos a+\cos b+\cos c=0}{=}(-\cos c)^2-\cos a\cos b\le\cos^2 c\ (\because \cos a\cos b\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\boxed{\text{Case 3i}}$ $\cos a\ge 0,\cos b\le 0$; $\cos c\ge 0$; LHS of $(*)=\cos^2 b+\cos a(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=}\cos^2 b-\cos a\cos c\le\cos^2 b\ (\because \cos a\cos c\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\boxed{\text{Case3ii}}$ $\cos a\ge 0,\cos b\le 0$, $\cos c\le 0$; LHS of $(*)=\cos^2 a+\cos b(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=}\cos^2 a-\cos b\cos c\le\cos^2 a\ (\because \cos b\cos c\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\boxed{\text{Case 4i}}$ $\cos a\le 0,\cos b\ge 0$; $\cos c\ge 0$; LHS of $(*)=\cos^2 a+\cos b(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=}\cos^2 a-\cos b\cos c\le\cos^2 a\ (\because \cos b\cos c\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\boxed{\text{Case4ii}}$ $\cos a\le 0,\cos b\ge 0$; $\cos c\le 0$; LHS of $(*)=\cos^2 b+\cos a(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=}\cos^2 b-\cos a\cos c\le\cos^2 b\ (\because \cos a\cos c\ge 0)\le 1$$

$$\Rightarrow (*)\text{ is true}$$

$\therefore$ combining *all* cases, $(*)$ is true $\forall\, a,b\in\mathbb{R}$ $\therefore \cos a+\cos b+\cos c=0$

$$\Rightarrow \cos^2 a+\cos^2 b+\cos^2 c\le 2\ \forall\, a,b,c\in\mathbb{R}\ (\text{QED})$$