ROMANIAN MATHEMATICAL MAGAZINE

If $x+1\neq 0$, then :

$$x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024}\geq 2$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{For } \alpha=-1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n-1-n\alpha=n-1\geq 0 \text{ and}$$
$$\forall\, \alpha>-1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n\geq 1+n\alpha$$
$$\therefore \forall\, \alpha\geq 1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n\geq 1+n\alpha \to (1)$$
$$\text{Now}, \forall\, x\in\mathbb{R}-\{-1\}, x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024}$$
$$=\left(1+(x^2-1)\right)^{1012}+\left(1+\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\right)^{2024}$$
$$\overset{\text{via }(1)}{\geq} 1+1012(x^2-1)+1+1012\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\overset{?}{\geq} 2$$
$$\Leftrightarrow 1012(x^2-1)+1012\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\overset{?}{\geq} 0 \Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2\overset{?}{\geq} 2$$
$$\Leftrightarrow x^2(x+1)^2+(x-3)^2\overset{?}{\geq} 2(x+1)^2 \Leftrightarrow x^4+2x^3-10x+7\overset{?}{\geq} 0$$
$$\Leftrightarrow (x-1)^2(x^2+4x+7)\overset{?}{\geq} 0 \Leftrightarrow (x-1)^2\left((x^2+4x+4)+3\right)\overset{?}{\geq} 0$$
$$\Leftrightarrow (x-1)^2((x+2)^2+3)\overset{?}{\geq} 0 \to \text{true} \therefore x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024}\geq 2$$
$$\forall\, x\in\mathbb{R}-\{-1\}, ''='' \text{ iff } x=1 \text{ (QED)}$$

Solution 2 by Eric Dimitrie Cismaru-Romania

Using Radon's Inequality, we have :

$$\frac{(x^2)^{1012}}{1^{1011}}+\frac{\left[\left(\frac{x-3}{x+1}\right)^2\right]^{1012}}{1^{1011}}\geq\frac{\left[x^2+\left(\frac{x-3}{x+1}\right)^2\right]^{1012}}{2^{1011}}\geq 2\Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2\geq 2\Leftrightarrow$$
$$\Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2\geq 2\Leftrightarrow x^2+\frac{x^2-6x+9}{x^2+2x+1}\geq 2\Leftrightarrow$$
$$\Leftrightarrow x^4+2x^3+2x^2-6x+9\geq 2x^2+4x+2\Leftrightarrow$$
$$\Leftrightarrow x^4+2x^3+7\geq 10x\Leftrightarrow x^4+2x^3-10x+7=x^4+3x^3-x^3-7x-3x+7=$$

$$= x^3(x-1) + 3x(x-1)(x+1) - 7(x-1) =$$

$$= (x-1)[x^3 + 3x^2 + 3x - 7] = (x-1)[(x+1)^3 - 2^3] =$$

$$= (x-1)^2[(x+1)^2 + 2(x+1) + 4] =$$

$$= (x-1)^2(x^2 + 4x + 7) = (x-1)^2[(x+2)^2 + 3] \geq 0,$$

so our inequality is proven. Equality holds iff $x = 1$ (If $(x+2)^2 + 3 = 0$, we would have $(x+2)^2 = -3 \geq 0$, impossible).