ROMANIAN MATHEMATICAL MAGAZINE

If $a \geq b \geq c > 0$, then prove that :

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

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Let $b = c + x\ (x \geq 0)$ and $a = b + y\ (y \geq 0) = c + x + y$ and then :

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

$$\Leftrightarrow (c+x+y)^2(c+x)y+(c+x)^2cx+c^2(c+x+y)(-x-y)\geq 0$$

$$\Leftrightarrow c^2x^2+c^2xy+c^2y^2+cx^3+3cx^2y+4cxy^2+cy^3+x^3y+2x^2y^2+xy^3\geq 0$$

$$\rightarrow \text{true} \because c>0 \text{ and } x,y\geq 0 \therefore a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

$$\forall\ a\geq b\geq c>0,\text{''}=\text{''}\ \text{iff } a=b=c\ (\text{QED})$$