ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

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$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

First, let's look at the solution integral ...

$$\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx\overset{I.B.P}{\triangleq}\left(-2x\sqrt{\cos(x)}\right)\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\sqrt{\cos(x)}\,dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos(x)}\,dx$$

$$\text{Substitution}:\ \cos(x)=t,-\sin(x)\,dx=dt,\qquad dx=-\frac{1}{\sqrt{1-t^2}}\ t[0;1]$$

$$\Omega=2\int_0^1\frac{\sqrt{t}}{\sqrt{1-t^2}}dt\underset{dt=\frac{dy}{2\sqrt{y}}}{\overset{2tdt=dy}{\triangleq}}\int_0^1\frac{\sqrt[4]{y}}{\sqrt{1-y}}\frac{dy}{\sqrt{y}}=\int_0^1 y^{\frac{1}{4}-\frac{1}{2}}(1-y)^{-\frac{1}{2}}dy$$

$$=\int_0^1 y^{-\frac{1}{4}}(1-y)^{-\frac{1}{2}}dy=$$

$$\beta\left(\frac{3}{4},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}$$

The Solution of our integral is completed, now let's return to the previous expression ...

$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma^2\left(\frac{3}{4}\right)}\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\Gamma^2\left(\frac{3}{4}\right)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{\sqrt{\pi}}{\frac{1}{4}\cdot\Gamma\left(1-\frac{1}{4}\right)\cdot\Gamma\left(1+\frac{1}{4}\right)}=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{4\sqrt{\pi}}{\sqrt{2}\,\pi}-3x=0$$

$$x^2-3x+2=0\ \to(x-1)(x-2)=0\to\ \textit{Answer}:\ x=1\ \textit{and}\ x=2$$