ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$\int_0^1 \frac{\ln(x)\big(1+\ln(x)+\ln(1-x)\big)}{1+x^2}dx = \frac{1}{64}\big(7\pi^3+4\pi ln^2(2)-64G-64Im\{Li_3(1+i)\}\big)$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \frac{\ln(x)\big(1+\ln(x)+\ln(1-x)\big)}{1+x^2}dx$$

$$= \int_0^1 \frac{\ln(x)}{1+x^2}dx + \int_0^1 \frac{ln^2(x)}{1+x^2}dx + \int_0^1 \frac{\ln(x)\ln(1-x)}{1+x^2}dx = \Omega_1+\Omega_2+\Omega_3$$

Note:

$$\left\{\int_0^1 \frac{\ln^k(x)}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln^k(x)\,dx = (-1)^k k!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{k+1}} = (-1)^k k!\,\beta(k+1)\right\}$$

$$\Omega_1 = \int_0^1 \frac{\ln(x)}{1+x^2}dx = -\beta(2) = -G;\ \Omega_2 = \int_0^1 \frac{ln^2(x)}{1+x^2}dx = 2\beta(3) = \frac{\pi^3}{16}$$

$$\Omega_3 = \int_0^1 \frac{\ln(x)\ln(1-x)}{1+x^2}dx,\ \Omega_3(a) = \int_0^1 \frac{\ln(x)\ln(1-ax)}{1+x^2}dx,\ \{\Omega_3(1)=\Omega_3, \Omega_3(0)=0\}$$

$$\frac{d}{da}\Omega_3(a) = \frac{d}{da}\int_0^1 \frac{\ln(x)\ln(1-ax)}{1+x^2}dx = \int_0^1 \frac{xln(x)}{(ax-1)(1+x^2)}dx =$$

$$= \frac{a}{1+a^2}\int_0^1 \frac{\ln(x)}{1+x^2}dx - \frac{1}{1+a^2}\int_0^1 \frac{xln(x)}{1+x^2}dx - \frac{a}{1+a^2}\int_0^1 \frac{\ln(x)}{1-ax}dx =$$

$$= -\frac{a\beta(2)}{1+a^2} + \frac{1}{1+a^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2} - \frac{a}{1+a^2}\sum_{n=0}^{\infty}a^n\int_0^1 x^n\ln(x)dx =$$

$$= -\frac{aG}{1+a^2} + \frac{\eta(2)}{4(1+a^2)} + \frac{a}{1+a^2}\sum_{n=0}^{\infty}\frac{a^n}{(n+1)^2} = -\frac{aG}{1+a^2} + \frac{\pi^2}{48(1+a^2)} + \frac{Li_2(a)}{1+a^2}$$

$$\Omega_3 = \frac{\pi^2}{48}\int_0^1 \frac{1}{1+a^2}da + \int_0^1 \frac{Li_2(a)}{1+a^2}da - G\int_0^1 \frac{a}{1+a^2}da = \frac{\pi^2}{48}I_1 + I_2 - GI_3$$

$$I_1 = \int_0^1 \frac{1}{1+a^2}da = [\arctan(a)]_0^1 = \arctan(1) = \frac{\pi}{4}$$

$$I_2 = \int_0^1 \frac{Li_2(a)}{1+a^2}da = \sum_{n=0}^{\infty}(-1)^n\int_0^1 a^{2n}Li_2(a)da$$

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Notes:

$$\begin{Bmatrix} \int_0^1 x^{n-1} Li_2(x)dx = \frac{\zeta(2)}{n} - \frac{H_n}{n^2} \\ \sum_{n=1}^{\infty} \frac{x^n}{n^2} H_n = Li_3(x) - Li_3(1-x) + \ln(1-x) Li_2(1-x) + \frac{1}{2}\ln(x) ln^2(1-x) + \zeta(3), \quad |x| \le 1 \end{Bmatrix}$$

$$I_2 = \sum_{n=0}^{\infty} (-1)^n \int_0^1 a^{2n+1-1} Li_2(a)da = \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2)}{2n+1} - \frac{H_{2n+1}}{(2n+1)^2}\right] =$$

$$= \zeta(2)\arctan(1) - \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}}{(2n+1)^2} = \frac{\pi^3}{24} - \text{İ}m\left\{\sum_{n=1}^{\infty} \frac{i^n H_n}{n^2}\right\}$$

Note: $\left\{\sum_{n=0}^{\infty} (-1)^n f(2n+1) = \text{İ}m\{\sum_{n=1}^{\infty} i^n f(n)\}\right\}$

$$I_2 = \frac{\pi^3}{24} - \text{İ}m\left\{Li_3(i) - Li_3(1-i) + \ln(1-i)\, Li_2(1-i) + \frac{1}{2}\ln(i)\, ln^2(1-i) + \zeta(3)\right\}$$

Notes:

$$\ln(1-i) = \ln\left(\sqrt{2}\left(cos\left(\frac{\pi}{4}\right) - isin\left(\frac{\pi}{4}\right)\right)\right) = \frac{1}{2}\ln(2) + \ln\left(e^{\frac{-i\pi}{4}}\right) = \frac{1}{2}\ln(2) - \frac{i\pi}{4}$$

$$\ln(i) = \ln\left(cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right) = \ln\left(e^{\frac{i\pi}{2}}\right) = \frac{i\pi}{2}$$

$$Li_2(1-i) = \frac{3}{8}\zeta(2) - i\left(\frac{\pi}{4}\ln(2) + G\right), Li_3(i) = \frac{i\pi^3}{32} - \frac{3\zeta(3)}{32}$$

$$I_2 = \frac{\pi^3}{24} + \frac{\pi}{16} ln^2(2) + \frac{G}{2}\ln(2) - \text{İ}m\{Li_3(1+i)\}$$

$$I_3 = \int_0^1 \frac{a}{1+a^2} da, \left\{\tan^{-1}(a) = t,\ dt = \frac{da}{1+a^2}, t\left[\frac{\pi}{4}, 0\right], a = \tan(t)\right\}$$

$$I_3 = \int_0^{\frac{\pi}{4}} \tan(t)dt = -[\ln(cost)]_0^{\frac{\pi}{4}} = \frac{\ln(2)}{2}$$

$$\Omega_3 = \frac{\pi^2}{48} I_1 + I_2 - GI_3 = \frac{\pi^3}{192} + \frac{\pi^3}{24} - \text{İ}m\{Li_3(1+i)\} + \frac{\pi}{16} ln^2(2) + \frac{G}{2}\ln(2) - \frac{G}{2}\ln(2) =$$

$$= \frac{3\pi^3}{64} + \frac{\pi}{16} ln^2(2) - \text{İ}m\{Li_3(1+i)\}$$

$$\int_0^1 \frac{\ln(x)(1+\ln(x)+\ln(1-x))}{1+x^2}dx = \Omega_1+\Omega_2+\Omega_3$$

$$= \frac{3\pi^3}{64}+\frac{\pi}{16}ln^2(2)-G-\mathrm{Im}\{Li_3(1+i)\}+\frac{\pi^3}{16}=$$

$$= \frac{1}{64}\left(7\pi^3+4\pi ln^2(2)-64G-64\mathrm{Im}\{Li_3(1+i)\}\right), \textit{Hence proved}$$