ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_{-1}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)\left(1+\frac{1-x}{1+x}e^x\right)} dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega = \int_{-1}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)\left(1+\frac{1-x}{1+x}e^x\right)} dx \underbrace{=}_{x\to -x} \int_{-1}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)\, e^x}{x(1+x^2)\left(e^x+\frac{1+x}{1-x}\right)} dx = I$$

$$2I = \int_{-1}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx$$

$$= \int_{-1}^{0} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx + \int_{0}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx = A + B =$$

$$A \underbrace{=}_{x\to -x} \int_{0}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx \quad \& \quad B = \int_{0}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx$$

$$2I = A + B = 2\int_{0}^{1} \frac{(\ln(x^2))^2 \tan^{-1}(x)}{x(1+x^2)} dx \to \frac{I}{4} = \int_{0}^{1} \frac{(\ln(x))^2 \tan^{-1}(x)}{x(1+x^2)} dx \Rightarrow \mathrm{I} = ?$$

$$\frac{I}{4} = \int_{0}^{1} \frac{(lnx)^2 \tan^{-1}(x)}{x} dx - \int_{0}^{1} \frac{x(lnx)^2 \tan^{-1}(x)}{1+x^2} dx = \Omega - \Psi$$

$$\Omega = \int_{0}^{1} \frac{(lnx)^2 \tan^{-1}(x)}{x} dx = \frac{\partial^2}{\partial a^2}\bigg|_{a=0} \int_{0}^{1} x^{a-1} \tan^{-1}(x)\, dx$$

$$= \frac{\partial^2}{\partial a^2}\bigg|_{a=0} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_{0}^{1} x^{a+2n}\, dx$$

$$\frac{\partial^2}{\partial a^2}\bigg|_{a=0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(a+2n+1)} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} = 2\beta(4)$$

$$\Psi = \int_{0}^{1} \frac{x(lnx)^2 \tan^{-1}(x)}{1+x^2} dx \underbrace{=}_{x\to\frac{1}{x}} \int_{1}^{\infty} \frac{(lnx)^2 \tan^{-1}\left(\frac{1}{x}\right)}{x(1+x^2)} dx$$

$$=\frac{\pi}{2}\int_1^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_1^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$=\frac{\pi}{2}\int_1^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_0^{1}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$\Psi=\frac{\pi}{2}\int_1^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_0^{1}\frac{(lnx)^2\tan^{-1}(x)}{x}dx-\Psi$$

$$2\Psi=\frac{\pi}{2}\int_1^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_0^{1}\frac{(lnx)^2\tan^{-1}(x)}{x}dx=M-N+\Omega$$

$$\Omega=2\beta(4)$$

$$M=\frac{\pi}{2}\int_1^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx\underbrace{=}_{x\to\frac{1}{x}}\frac{\pi}{2}\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+a+1}\,dx=$$

$$=\frac{\pi}{2}\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\frac{(-1)^n}{a+2n+2}=-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n)^3}=-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{-3}{4n^3}=\frac{3\pi\zeta(3)}{32}=M$$

$$N=\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx\Rightarrow f(a)=\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}dx\Rightarrow f(1)=N$$

$$(\text{Feynman trick}\}\to\frac{d}{da}f(a)=\frac{d}{da}\int_0^{\infty}\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}dx$$

$$=\int_0^{\infty}\frac{\partial}{\partial a}\left(\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}\right)dx$$

$$\frac{d}{da}f(a)=\int_0^{\infty}\frac{(lnx)^2}{(1+(ax)^2)(1+x^2)}dx=\frac{a^2}{a^2-1}\int_0^{\infty}\frac{(lnx)^2}{1+(ax)^2}dx-\frac{1}{a^2-1}\int_0^{\infty}\frac{(lnx)^2}{1+x^2}dx$$

$$=K-H$$

$$K=\frac{a^2}{a^2-1}\int_0^{\infty}\frac{(lnx)^2}{1+(ax)^2}dx\underbrace{=}_{ax\to x}\frac{a}{a^2-1}\int_0^{\infty}\frac{\left(\ln\left(\frac{x}{a}\right)\right)^2}{1+x^2}dx=$$

$$\frac{a}{a^2-1}\int_0^{\infty}\frac{(lnx)^2}{1+x^2}dx-\frac{2aln(a)}{a^2-1}\int_0^{\infty}\frac{\ln(x)}{1+x^2}dx+\frac{a(lna)^2}{a^2-1}\int_0^{\infty}\frac{1}{x^2+1}dx$$

$$\{note:\int_0^{\infty}\frac{(lnx)^2}{1+x^2}dx=\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\int_0^{\frac{\pi}{2}}(\tan x)^a\,dx=\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\frac{1}{2}\beta(\frac{a+1}{2}\cdot\frac{1-a}{2}=\frac{\pi^3}{8}\}$$

$$\left\{ note: \int_0^{\infty} \frac{ln(x)}{1+x^2} dx = \frac{\partial}{\partial a}\Big|_{a=0} \int_0^{\frac{\pi}{2}} (\tan x)^a\, dx = \frac{\partial}{\partial a}\Big|_{a=0} \frac{1}{2}\beta(\frac{a+1}{2} \cdot \frac{1-a}{2} = 0 \right\}$$

$$H = \frac{1}{a^2-1}\int_0^{\infty} \frac{(lnx)^2}{1+x^2} dx = \frac{\pi^3}{8(a^2-1)}$$

$$\underbrace{=}_{f(0)=0\ \&\ f(1)=N} f(1) = \int_0^1 df(a) = \int_0^1 (K-H)da = \frac{\pi^3}{8}\int_0^1 \frac{da}{a+1} + \frac{\pi}{2}\int_0^1 \frac{a(lna)^2}{a^2-1} da$$

$$N = f(1) = \frac{\pi^3}{8}\ln(2) - \frac{\pi}{2}\frac{\partial^2}{\partial b^2}\Big|_{b=0} \sum_{n=0}^{\infty} \int_0^1 a^{2n+b+1}\, da$$

$$\frac{\pi^3}{8}\ln(2) - \frac{\pi}{2}\frac{\partial^2}{\partial b^2}\Big|_{b=0} \sum_{n=0}^{\infty} \frac{1}{2n+b+2} = \frac{\pi^3}{8}\ln(2) - \pi\sum_{n=1}^{\infty}\frac{1}{8n^3} = \frac{\pi^3}{8}\ln(2) - \frac{\pi\zeta(3)}{8} = N$$

$$I = 4\Omega - 4\Psi = 4\Omega - 4\left(\frac{M}{2} - \frac{N}{2} + \frac{\Omega}{2}\right) = 2\Omega - 2M + 2N$$

$$I = 4\beta(4) - \frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{4} ln(2) - \frac{\pi}{4}\zeta(3)$$

$$\text{ANSWER} = I = \frac{\psi^{(3)}(\frac{1}{4})}{192} - \frac{\pi^4}{24} - \frac{7\pi}{16}\zeta(3) + \frac{\pi^3}{4} ln(2)$$

{note section}

$$\beta(4) = \frac{\psi^{(3)}(\frac{1}{4})}{768} - \frac{8\pi^4}{768}$$